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NOTE ON STAR-AUTONOMOUS COMONADS

CRAIG PASTRO

ABSTRACT. We develop an alternative approach to star-autonomous comonads via linearly distributive categories. It is shown that in the autonomous case the notions of star-autonomous comonad and Hopf comonad coincide.

1. Introduction

Given a linearly distributive category $\mathcal{C}$, this note determines what structure is required of a comonad G on $\mathcal{C}$ so that $\mathcal{C}^G$, the category of Eilenberg-Moore coalgebras of G, is again a linearly distributive category. Furthermore, if $\mathcal{C}$ is equipped with negations (and is hence a star-autonomous category), the structure required to lift the negations to $\mathcal{C}^G$ is determined as well. This latter is equivalent to lifting star-autonomy and it is shown that the notion presented is equivalent to a star-autonomous comonad [PS09]. As a consequence of the presentation given here, it may be easily seen that any star-autonomous comonad on an autonomous category is a Hopf monad [BV07].

2. Lifting linear distributivity

Suppose $\mathcal{C}$ is a monoidal category and $G : \mathcal{C} \to \mathcal{C}$ is a comonad on $\mathcal{C}$. Recall that $\mathcal{C}^G$, the category of (Eilenberg-Moore) coalgebras of G, is monoidal if and only if G is a monoidal comonad [M02]. In this section we are interested in the structure required to lift linear distributivity to the category of coalgebras.

A linearly distributive category $\mathcal{C}$ is a category equipped with two monoidal structures $(\mathcal{C}, \star, I)$ and $(\mathcal{C}, \diamond, J)$,[1] and two compatibility natural transformations (called "linear distributions")

$$\partial_l : A \star (B \diamond C) \to (A \star B) \diamond C$$
$$\partial_r : (B \diamond C) \star A \to B \diamond (C \star A),$$

satisfying a large number of coherence diagrams [CS97].

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[1]For simplicity we assume that the monoidal structures are strict, although this is not necessary. Furthermore, in their original paper [CS97] the tensor products $\star$ and $\diamond$ are respectively denoted by $\otimes$ and $\oplus$, and called *tensor* and *par*, emphasizing their connection to linear logic.

Suppose $G = (G, \delta, \epsilon)$ is a comonad on a linearly distributive category $\mathcal{C}$ which is a monoidal comonad on $\mathcal{C}$ with respect to both $\star$ and $\diamond$, with structure maps (G, ϕ, ϕ_0) and (G, ψ, ψ_0) respectively. If, for G-coalgebras A, B, and C, the comonad G satisfies

$$\begin{tikzcd}
GA \star (GB \diamond GC) \arrow[r, "1\star\psi"] \arrow[d, "\partial_l"'] & GA \star G(B \diamond C) \arrow[r, "\phi"] & G(A \star (B \diamond C)) \arrow[d, "\partial_l"] \\
(GA \star GB) \diamond GC \arrow[r, "\phi\diamond 1"] & G(A \star B) \diamond GC \arrow[r, "\psi"] & G((A \star B) \diamond C),
\end{tikzcd} \tag{1}$$

it may be seen that the morphism ∂_l becomes a G-coalgebra morphism. If G satisfies a similar axiom for ∂_r, i.e.,

$$\begin{tikzcd}
(GB \diamond GC) \star GA \arrow[r, "\psi\star 1"] \arrow[d, "\partial_r"'] & G(B \diamond C) \star GA \arrow[r, "\phi"] & G((B \diamond C) \star A) \arrow[d, "\partial_r"] \\
GB \diamond (GC \star GA) \arrow[r, "1\diamond\phi"] & GB \diamond G(C \star A) \arrow[r, "\psi"] & G(B \diamond (C \star A)),
\end{tikzcd} \tag{2}$$

then ∂_r also becomes a G-coalgebra morphism. Thus,

2.1. Proposition. *Given a linearly distributive category $\mathcal{C}$ and a comonad $G : \mathcal{C} \to \mathcal{C}$ satisfying axioms* (1) *and* (2)*, the category $\mathcal{C}^G$ is a linearly distributive category.*

2.2. Example. Let $\mathcal{C}$ be a symmetric linearly distributive category and $(B, \mu, \eta, \delta, \epsilon)$ a bialgebra in $\mathcal{C}$ with respect to $\diamond$. That is, the structure morphisms are given as

$$\begin{aligned} &\mu : B \diamond B \to B \qquad && \delta : B \to B \diamond B \\ &\eta : J \to B && \epsilon : B \to J. \end{aligned}$$

Then, $G = B \diamond -$ is a comonad and is monoidal with respect to both $\diamond$ and $\star$. The latter via $I \cong J \diamond I \xrightarrow{\eta\diamond 1} B \diamond I$, and the following,

$$\begin{aligned}
(B \diamond U) \star (B \diamond V) &\xrightarrow{\ \partial_r\ } B \diamond (U \star (B \diamond V)) \\
&\xrightarrow{1 \diamond (1 \star c)} B \diamond (U \star (V \diamond B)) \\
&\xrightarrow{1 \diamond \partial_l} B \diamond ((U \star V) \diamond B) \\
&\xrightarrow{1 \diamond c} B \diamond (B \diamond (U \star V)) \\
&\xrightarrow{\ \cong\ } (B \diamond B) \diamond (U \star V) \\
&\xrightarrow{\mu \star 1} B \diamond (U \star V).
\end{aligned}$$

Rather large diagrams, which we leave to the faith of the reader, prove that $B \diamond -$ satisfies (1) and (2), so that $\mathcal{C}^B = \mathbf{Comod}_{\mathcal{C}}(B)$, the category of comodules of B, is a linearly distributive category.

3. Lifting negations

Suppose now that $\mathcal{C}$ is a linearly distributive category equipped with negations S and S' (corresponding to ${}^{\perp}(-)$ and $(-)^{\perp}$ in [CS97]). That is, functors $S, S' : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ together with the following (dinatural) evaluation and coevaluation morphisms

$$\begin{array}{ll} SA \star A \xrightarrow{e_A} J & \qquad A \star S'A \xrightarrow{e'_A} J \\ I \xrightarrow{n_A} A \diamond SA & \qquad I \xrightarrow{n'_A} S'A \diamond A, \end{array} \tag{3}$$

satisfying the four evident "triangle identities". One such is

$$\Big(A \cong I \star A \xrightarrow{n \star 1} (A \diamond SA) \star A \xrightarrow{\partial_r} A \diamond (SA \star A) \xrightarrow{1 \diamond e} A \diamond J \cong A\Big) = 1_A.$$

If $\mathcal{C}$ is equipped with such negations we say simply that $\mathcal{C}$ is a *linearly distributive category with negations.*

We are interested to lift negations to $\mathcal{C}^G$. This means we must ensure that the "negation" functors $S, S' : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ lift to functors $(\mathcal{C}^G)^{\mathrm{op}} \to \mathcal{C}^G$, and the evaluation and coevaluation morphisms are in $\mathcal{C}^G$, i.e., are G-coalgebra morphisms.

The following is essentially known from [S72].

3.1. Proposition. *A (contravariant) functor $S : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ may be lifted to a functor $\widetilde{S} : (\mathcal{C}^G)^{\mathrm{op}} \to \mathcal{C}^G$ such that the diagram*

$$\begin{tikzcd} (\mathcal{C}^G)^{\mathrm{op}} \arrow[r, "\widetilde{S}"] \arrow[d, "U"'] & \mathcal{C}^G \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}} \arrow[r, "S"] & \mathcal{C}, \end{tikzcd}$$

(in which $U : \mathcal{C}^G \to \mathcal{C}$ is the underlying functor) commutes, if and only if there is a natural transformation

$$\nu : S \to GSG$$

satisfying the following two axioms

$$\begin{tikzcd} S \arrow[r, "\nu"] \arrow[dr, "S\epsilon"'] & GSG \arrow[d, "\epsilon_{SG}"] \\ & SG \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\nu"] \arrow[dr, "\nu"'] & GSG \arrow[r, "\delta_{SG}"] & G^2SG \\ & GSG \arrow[r, "G\nu_G"] & G^2SG^2. \arrow[u, "G^2S\delta"'] \end{tikzcd} \tag{4}$$

This may be viewed as a distributive law of a contravariant functor over a comonad [S72]. In this case, we say that S *may be lifted to* $\mathcal{C}^G$, and a functor $\widetilde{S} : (\mathcal{C}^G)^{\mathrm{op}} \to \mathcal{C}^G$ is defined as

$$\widetilde{S}(A, \gamma) = \Big(SA,\ SA \xrightarrow{\nu} GSGA \xrightarrow{GS\gamma} GSA\Big) \qquad \widetilde{S}(f) = Sf.$$

(To see the reverse direction, suppose (A,γ) is a coalgebra and $\widetilde{S}$ is a functor $\mathcal{C}^G \to \mathcal{C}^G$, so that $\widetilde{S}A = (SA, \widetilde{\gamma})$ is again a coalgebra. Define

$$\nu := SA \xrightarrow{\widetilde{\gamma}} GSA \xrightarrow{GS\epsilon_A} GSGA,$$

which may be seen to satisfy the axioms in (4).) We will usually let the context differentiate between S and $\widetilde{S}$ and simply write S in both cases.

Now, suppose S and S' may be lifted to $\mathcal{C}^G$, that is, they are equipped respectively with natural transformations

$$\nu : S \to GSG \qquad \text{and} \qquad \nu' : S' \to GS'G,$$

satisfying (4). It remains to lift the evaluation and coevaluation morphisms (3). Consider the following axioms.

$$\begin{tikzcd}
SA \star GA \arrow[r, "1\star\epsilon"] \arrow[d, "\nu\star\delta"'] & SA \star A \arrow[r, "e_A"] & J \arrow[d, "\psi_0"] \\
GSGA \star G^2A \arrow[r, "\phi"] & G(SGA \star GA) \arrow[r, "Ge_{GA}"] & GJ
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[d, "n"'] & GI \arrow[r, "Gn"] & G(A \diamond SA) \arrow[r, "G(1\diamond S\epsilon)"] & G(A \diamond SGA) \\
GA \diamond SGA \arrow[r, "1\diamond\nu"] & GA \diamond GSG^2A \arrow[rr, "\phi"] & & G(A \diamond SG^2A) \arrow[u, "G(1\diamond S\delta)"']
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
GA \star S'A \arrow[r, "\epsilon\star 1"] \arrow[d, "\delta\star\nu'"'] & A \star S'A \arrow[r, "e'_A"] & J \arrow[d, "\psi_0"] \\
G^2A \star GS'GA \arrow[r, "\phi"] & G(GA \star S'GA) \arrow[r, "Ge'_{GA}"] & GJ
\end{tikzcd} \tag{7}$$

$$\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[d, "n'"'] & GI \arrow[r, "Gn'"] & G(S'A \diamond A) \arrow[r, "G(S'\epsilon\diamond 1)"] & G(S'GA \diamond A) \\
S'GA \diamond GA \arrow[r, "\nu'\diamond 1"] & GS'G^2A \diamond GA \arrow[rr, "\phi"] & & G(S'G^2A \diamond A) \arrow[u, "G(S'\delta\diamond 1)"']
\end{tikzcd} \tag{8}$$

3.2. Proposition. *Suppose $\mathcal{C}$ is a linearly distributive category with negation, G is a monoidal comonad satisfying axioms* (1) *and* (2) *(so that $\mathcal{C}^G$ is linearly distributive), and that S and S' may be lifted to $\mathcal{C}^G$. Then, G satisfies axioms* (5), (6), (7), *and* (8) *if and only if $\mathcal{C}^G$ is a linearly distributive category with negation.*

PROOF. Suppose (A,γ) is a G-coalgebra. We start by proving that axiom (5) holds if and only if $e : SA \star A \to J$ is a G-coalgebra morphism. The following diagram proves the "only if" direction,

$$\begin{tikzcd}
SA \star A \arrow[r, "\nu\star\gamma"] \arrow[dd, "1\star\gamma"] \arrow[dddd, bend right, "1"'] & GSGA \star GA \arrow[r, "\phi"] \arrow[dd, "1\star G\gamma"] & G(SGA \star A) \arrow[dr, "G(S\gamma\star 1)"] \arrow[dd, "G(1\star\gamma)"'] & {} \\
{} & {} & {} & G(SA \star A) \arrow[dd, "Ge"] \\
SA \star GA \arrow[r, "\nu\star\delta"] \arrow[dd, "1\star\epsilon"] & GSGA \star G^2A \arrow[r, "\phi"] \arrow[d, phantom, "(5)"] & G(SGA \star GA) \arrow[dr, "Ge"] & {} \\
{} & {} & {} & GJ, \\
SA \star A \arrow[r, "e"] & J \arrow[urr, "\psi_0"] & {} & {}
\end{tikzcd}$$

and this next diagram the "if" direction

$$\begin{tikzcd}
{} & SA \star GA \arrow[dl, "1\star\epsilon"'] \arrow[d, "S\epsilon\star 1"] \arrow[r, "\nu\star\delta"] & GSGA \star G^2A \arrow[d, "GSG\epsilon\star 1"'] \arrow[dr, "1"] \arrow[drr, bend left, "\phi"] & {} & {} \\
SA \star A \arrow[dr, "e"'] & SGA \star GA \arrow[r, "\nu\star\delta"] \arrow[d, "e_{GA}"] & GSG^2A \star G^2A \arrow[r, "GS\delta\star 1"] & GSGA \star G^2A \arrow[r, "\phi"] & G(SGA \star GA) \arrow[d, "Ge_{GA}"] \\
{} & J \arrow[rrr, "\psi_0"] & {} & {} & GJ,
\end{tikzcd}$$

where the bottom square commutes as e_{GA} is a G-coalgebra morphism.

Next we prove that axiom (6) holds if and only if $n : I \to A \diamond SA$ is a G-coalgebra morphism. The "only if" direction is given by

$$\begin{tikzcd}
I \arrow[rr, "\phi_0"] \arrow[d, "n"'] \arrow[dr, "n"] & {} & GI \arrow[rr, "Gn"] \arrow[d, phantom, "(6)"] & {} & G(A \diamond SA) \arrow[d, "G(1\diamond S\epsilon)"'] \arrow[dd, bend left, "1"] \\
A \diamond SA \arrow[dr, "\gamma\diamond 1"'] & GA \diamond SGA \arrow[r, "1\diamond\nu"] \arrow[d, "1\diamond S\gamma"] & GA \diamond GSG^2A \arrow[r, "\phi"] \arrow[d, "1\diamond GSG\gamma"'] & G(A \diamond SG^2A) \arrow[r, "G(1\diamond S\delta)"] \arrow[d, "G(1\diamond SG\gamma)"'] & G(A \diamond SGA) \arrow[d, "G(1\diamond S\gamma)"'] \\
{} & GA \diamond SA \arrow[r, "1\diamond\nu"] & GA \diamond GSGA \arrow[r, "\phi"] & G(A \diamond SGA) \arrow[r, "G(1\diamond S\gamma)"] & G(A \diamond SA),
\end{tikzcd}$$

and the "if" direction by

$$
\begin{tikzcd}
I \arrow[rrrr, "\phi_0"] \arrow[d, "n_{GA}"'] & & & & GI \arrow[d, "Gn_{GA}"'] \arrow[dr, "Gn"] & \\
GA \diamond SGA \arrow[r, "\delta\diamond\nu"] \arrow[dr, "1\diamond\nu"'] & G^2A \diamond GSG^2A \arrow[r, "\psi"] \arrow[d, "G\epsilon\diamond 1"] & G(GA \diamond SG^2A) \arrow[rr, "G(1\diamond S\delta)"] \arrow[d, "G(\epsilon\diamond 1)"] & & G(GA \diamond SGA) \arrow[d, "G(\epsilon\diamond 1)"'] & G(A \diamond SA) \arrow[dl, "G(1\diamond S\epsilon)"] \\
& GA \diamond GSG^2A \arrow[r, "\psi"] & G(A \diamond SG^2A) \arrow[rr, "G(1\diamond S\delta)"] & & G(A \diamond SGA), &
\end{tikzcd}
$$

where the top square commutes as n_{GA} is a G-coalgebra morphism.

The remaining two axioms are proved similarly. ∎

4. Star-autonomous comonads

Suppose $\mathcal{C} = (\mathcal{C}, \otimes, I, S, S')$ is a star-autonomous category. A star-autonomous comonad $G : \mathcal{C} \to \mathcal{C}$ is a comonad satisfying axioms (described below) so that $\mathcal{C}^G$ becomes a star-autonomous category [PS09]. In this section we show that comonads as in Proposition 3.2 and star-autonomous comonads coincide.

We recall the definition of star-autonomous comonad [PS09], but, as it suits our needs better here, we present a more symmetric version. First recall that a star-autonomous category $\mathcal{C} = (\mathcal{C}, \otimes, I, S, S')$ may be defined as a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with an adjoint equivalence

$$S \dashv S' : \mathcal{C}^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{C}$$

such that

$$\mathcal{C}(A \otimes B, SC) \cong \mathcal{C}(A, S(B \otimes C)), \tag{9}$$

natural in $A, B, C \in \mathcal{C}$. The functor S is called the *left star operation* and S' the *right star operation*.

By the Yoneda lemma, the isomorphism in (9) determines, and is determined by, the two following "evaluation" morphisms:

$$e = e_{A,B} : S(A \otimes B) \otimes A \to SB \quad \text{and} \quad e' = e'_{B,A} : B \otimes S'(A \otimes B) \to S'A.$$

4.1. Definition. *A* star-autonomous comonad *on a star-autonomous category $\mathcal{C}$ is a monoidal comonad $G : \mathcal{C} \to \mathcal{C}$ equipped with*

$$\nu : S \to GSG \qquad \textit{and} \qquad \nu' : S' \to GS'G,$$

satisfying (4) *(i.e., S and S' may be lifted to $\mathcal{C}^G$), and this data must be such that the following four diagrams commute.*

$$\begin{tikzcd}
SS'G \arrow[r, "\cong"] \arrow[d, "\nu"'] & G \arrow[d, "\cong"] \\
GSGS'G \arrow[r, "GS\nu'"] & GSS'
\end{tikzcd}
\qquad
\begin{tikzcd}
S'SG \arrow[r, "\cong"] \arrow[d, "\nu'"'] & G \arrow[d, "\cong"] \\
GS'GSG \arrow[r, "GS\nu"] & GS'S
\end{tikzcd}$$

$$\begin{tikzcd}
& S(A\otimes B)\otimes GA \arrow[r, "1\otimes\epsilon"] \arrow[dl, "\nu\otimes\delta"'] & S(A\otimes B)\otimes A \arrow[r, "e_{A,B}"] & SB \arrow[dr, "\nu"] & \\
GSG(A\otimes B)\otimes G^2A \arrow[dr, "\phi"'] & & & & GSGB \\
& G(SG(A\otimes B)\otimes GA) \arrow[rr, "G(S\phi\otimes 1)"'] & & G(S(GA\otimes GB)\otimes GA) \arrow[ur, "Ge_{GA,GB}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& GB\otimes S'(A\otimes B) \arrow[r, "\epsilon\otimes 1"] \arrow[dl, "\delta\otimes\nu'"'] & B\otimes S'(A\otimes B) \arrow[r, "e'_{B,A}"] & S'A \arrow[dr, "\nu'"] & \\
G^2B\otimes GS'G(A\otimes B) \arrow[dr, "\phi"'] & & & & GS'GA \\
& G(GB\otimes S'G(A\otimes B)) \arrow[rr, "G(1\otimes S'\phi)"'] & & G(GB\otimes S'(GA\otimes GB)) \arrow[ur, "Ge'_{GB,GA}"'] &
\end{tikzcd}$$

The first two diagrams above ensure that the unit and the counit of the adjoint equivalence $S \dashv S'$ lifts to $\mathcal{C}^G$, while the latter two diagrams above respectively ensure that e and e' are G-coalgebra morphisms, so that the isomorphism (9) also lifts to $\mathcal{C}^G$.

We wish to show that star-autonomous comonads and comonads as in Proposition 3.2 coincide. It should not be surprising considering the following theorem.

4.2. Theorem. [CS97, Theorem 4.5] *The notions of linearly distributive categories with negation and star-autonomous categories coincide.*

Given a star-autonomous category $(\mathcal{C}, \otimes, I_\otimes, S, S')$, identifying $\star := \otimes$ (and the monoidal unit $I := I_\star = I_\otimes$) and defining

$$A \diamond B := S'(SB \star SA) \cong S(S'B \star S'A) \qquad J := SI \cong S'I \tag{10}$$

results in a linearly distributive category with negations $(\mathcal{C}, \star, I, \diamond, J, S, S')$ [CS97]. In [CS97], they consider the symmetric case, but the correspondence between linearly distributive categories with negation and star-autonomous categories holds in the noncommutative case as well.

Given Theorem 4.2, Proposition 3.2 says that if $\mathcal{C}$ is star-autonomous, and G is such a comonad, then $\mathcal{C}^G$ is star-autonomous. We now compare the two definitions.

Suppose now that G is a comonad on a linearly distributive category $\mathcal{C}$, as in Proposition 3.2. We wish to show that it is a star-autonomous comonad. Rather than proving the axioms, it is simpler to show directly that the morphisms under consideration are G-coalgebra morphisms. To this end, the unit and the counit of the adjoint equivalence $S \dashv S'$ are defined respectively by the composites

$$A \cong I \star A \xrightarrow{n'_{SA}\star 1} (S'SA \diamond SA) \star A \xrightarrow{\partial_r} S'SA \diamond (SA \star A) \xrightarrow{1\diamond n} S'SA \diamond J \cong S'SA$$

and

$$S'SA \cong I \star S'SA \xrightarrow{n_A \star 1} (A \diamond SA) \star S'SA \xrightarrow{\partial_r} A \diamond (SA \star S'SA) \xrightarrow{1\diamond e'_{SA}} A \diamond J \cong A,$$

and the evaluation morphisms $e_{A,B}$ and $e'_{B,A}$ respectively by the composites

$$\begin{tikzcd}
S(A\star B)\star A \arrow[rr, dotted, "e_{A,B}"] \arrow[d, "\cong"'] & & SB \\
(S(A\star B)\star A)\star I \arrow[d, "1\star n"'] & & J\diamond SB \arrow[u, "\cong"'] \\
(S(A\star B)\star A)\star(B\diamond SB) \arrow[rr, "\partial_l"] & & (S(A\star B)\star A\star B)\diamond SB, \arrow[u, "e_{A\star B}\diamond 1"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
B\star S'(A\star B) \arrow[rr, dotted, "e'_{B,A}"] \arrow[d, "\cong"'] & & SB \\
I\star(B\star S'(A\star B)) \arrow[d, "n'\star 1"'] & & S'A\diamond J \arrow[u, "\cong"'] \\
(S'A\diamond A)\star(B\star S'(A\star B)) \arrow[rr, "\partial_r"] & & S'A\diamond(A\star B\star S'(A\star B)). \arrow[u, "1\diamond e'_{A\star B}"']
\end{tikzcd}$$

In the situation of Proposition 3.2, we see that all four of these morphisms are given as composites of G-coalgebra morphisms, and thus, are G-coalgebra morphisms themselves. Therefore, G is a star-autonomous comonad.

In the other direction suppose G is a star-autonomous comonad on a star-autonomous category $\mathcal{C}$. It is similar to show that it is a comonad satisfying the requirements of Proposition 3.2. Using the identifications in (10), the two linear distributions are defined as the following composites.

$$
\begin{tikzcd}
A \star (B \diamond C) \arrow[rr, dotted, "\partial_l"] \arrow[d, "\cong"'] & & (A \star B) \diamond C \\
A \otimes S'(SC \otimes SB) \arrow[dr, "1 \otimes S'(1 \otimes e)"'] & & S'(SC \otimes S(A \otimes B)) \arrow[u, "\cong"'] \\
& A \otimes S'(SC \otimes S(A \otimes B) \otimes A) \arrow[ur, "e'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(B \diamond C) \star A \arrow[rr, dotted, "\partial_r"] \arrow[d, "\cong"'] & & B \diamond (C \star A) \\
S(S'C \otimes S'B) \otimes A \arrow[dr, "S(e' \otimes 1) \otimes 1"'] & & S(S'(C \otimes A) \otimes S'B) \arrow[u, "\cong"'] \\
& S(A \otimes S'(C \otimes A) \otimes S'B) \otimes A) \arrow[ur, "e"'] &
\end{tikzcd}
$$

The evaluation maps e_A and e'_A are defined as $e_{A,I}$ and $e'_{A,I}$, and the coevaluation maps n_A and n'_A as

$$n_A = \left(I \cong SS'I \xrightarrow{S(e'_{A,I})} S(A \otimes S'A) = A \diamond SA \right)$$

$$n'_A = \left(I \cong S'SI \xrightarrow{S'(e_{A,I})} S'(SA \otimes A) = S'A \diamond A \right)$$

Again, each morphism is a G-coalgebra morphism, or composite thereof, and therefore is itself a G-coalgebra morphism.

Thus, both notions coincide, and we will simply call either a *star-autonomous comonad*, and let context differentiate the axiomatization.

4.3. Example. Any Hopf algebra H in a star-autonomous category $\mathcal{C}$ gives rise to a star-autonomous comonad $H \otimes - : \mathcal{C} \to \mathcal{C}$. See [PS09, p. 3515] for details.

4.4. Example. If $\mathcal{C}$ is a symmetric closed monoidal category with finite products, then we may apply an instance of the Chu construction [B79] to produce a star-autonomous category $\mathrm{Chu}(\mathcal{C}, 1)$. The category $\mathcal{C}$ fully faithfully embeds into $\mathrm{Chu}(\mathcal{C}, 1)$,

$$\mathcal{C} \hookrightarrow \mathrm{Chu}(\mathcal{C}, 1)$$

and this functor is strong symmetric monoidal. Thus, any Hopf algebra in $\mathcal{C}$ becomes a Hopf algebra in $\mathrm{Chu}(\mathcal{C}, 1)$, and thus, an example of a star-autonomous comonad.

5. The compact case $\star = \diamond$

If $\mathcal{C}$ is a linearly distributive category with negation for which $\star = \diamond$ (and thus, $I = J$), then $\mathcal{C}$ is an autonomous (= rigid) category. The functor S provides left duals, while S' provides right duals. It is not hard to see that, in this case, any star-autonomous monad G (after turning arrows around) is a Hopf monad [BV07]. Set $\star = \diamond$ and $I = J$ and dualize axioms (5), (6), (7), and (8). They correspond in [BV07] to axioms (23), (22), (21), and (20) respectively. (In their notation ${}^\vee(-) = S$ and $(-)^\vee = S'$.) Therefore, we have:

5.1. Proposition. *Star-autonomous monads on autonomous categories are Hopf monads.*

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Department of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
Email: `craig@math.kyushu-u.ac.jp`

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