Theory and Applications of Categories, Vol. 26, No. 28, 2012, pp. 768–787.

DÉCALAGE AND KAN'S SIMPLICIAL LOOP GROUP FUNCTOR

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ABSTRACT. Given a bisimplicial set, there are two ways to extract from it a simplicial set: the diagonal simplicial set and the less well known total simplicial set of Artin and Mazur. There is a natural comparison map between these simplicial sets, and it is a theorem due to Cegarra and Remedios and independently Joyal and Tierney, that this comparison map is a weak homotopy equivalence for any bisimplicial set. In this paper we will give a new, elementary proof of this result. As an application, we will revisit Kan's simplicial loop group functor G. We will give a simple formula for this functor, which is based on a factorization, due to Duskin, of Eilenberg and Mac Lane's classifying complex functor $\overline{W}$. We will give a new, short, proof of Kan's result that the unit map for the adjunction $G \dashv \overline{W}$ is a weak homotopy equivalence for reduced simplicial sets.

1. Introduction

The aim of this paper is to give new and hopefully simpler proofs of two theorems in the theory of simplicial sets, the first being a generalization to simplicial sets of Dold-Puppe's version [Dold-Puppe, 1961] of the Eilenberg-Zilber theorem from homological algebra, the second being an old result of Kan's on simplicial loop groups. This first result is due to Cegarra and Remedios and independently to Joyal and Tierney.

Recall that if X is a bisimplicial set, then the diagonal dX of X is the simplicial set obtained by precomposing the functor $X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$ with the opposite of the functor $\delta\colon \Delta \to \Delta \times \Delta$ given by $\delta([n]) = ([n], [n])$, i.e. $dX = X\delta$, so that the set of n-simplices of dX is $(dX)_n = X_{n,n}$. In particular, if A is a bisimplicial abelian group, then dA is a simplicial abelian group which we may regard as a chain complex via the Moore complex functor.

Alternatively, we could first form a bicomplex from A using the Moore bicomplex construction [Goerss-Jardine, 1999] and then forming the associated total complex, which we will denote $\operatorname{Tot} A$. There is a natural comparison map $dA \to \operatorname{Tot} A$ and the generalized Eilenberg-Zilber theorem [Dold-Puppe, 1961, Goerss-Jardine, 1999] says that this comparison map is a chain homotopy equivalence.

There is a generalization of this comparison with bisimplicial abelian groups replaced by bisimplicial sets. In this case the total complex construction is replaced by what is

The author was supported by the EPSRC [grant number EP/I010610/1]
Received by the editors 2012-03-02 and, in revised form, 2012-12-11.
Transmitted by Myles Tierney. Published on 2012-12-13.
2010 Mathematics Subject Classification: 18G30, 55U10.
Key words and phrases: simplicial loop group, décalage, Artin-Mazur total simplicial set.

variously known as the *total simplicial set* or *Artin-Mazur codiagonal* TX of a simplicial set X (see [Artin-Mazur, 1966]). This construction extends to define a functor $T\colon \mathbf{SS} \to \mathbf{S}$ from the category $\mathbf{SS}$ of bisimplicial sets to the category $\mathbf{S}$ of simplicial sets.

The construction TX is the analog for simplicial sets of the process of forming the total complex $\operatorname{Tot} C$ of a double complex C. In fact, if

$$N\colon s\mathbf{Ab} \leftrightarrows \mathbf{Ch}_{\geq 0}\colon \Gamma$$

denotes the Dold-Kan correspondence, then for any bisimplicial abelian group A there is an isomorphism between NTA and $\operatorname{Tot} NA$ (see [Cegarra-Remedios, 2005]).

As mentioned above, the Eilenberg-Zilber theorem in homological algebra has a generalization for simplicial sets. For any bisimplicial set X, there is a natural comparison map

$$dX \to TX$$

between the diagonal simplicial set of X and the total simplicial set of X. We have the following result.

1.1. Theorem. [Cegarra-Remedios, 2005, Joyal-Tierney, 2011] *Let X be a bisimplicial set. Then the comparison map $dX \to TX$ is a weak homotopy equivalence.*

The first published proof of this result was given in [Cegarra-Remedios, 2005], with the authors noting that this fact is stated without proof in [Cordier, 1987] where it is attributed to Zisman (unpublished). When X is a bisimplicial group, a closely related result was proven by Quillen [Quillen, 1966]. The proof of Theorem 1.1 given in [Cegarra-Remedios, 2005] is unfortunately somewhat complicated, and so it is of interest to have a simpler proof. One such proof, incorporating some ideas of Cisinski, is given by Joyal and Tierney in their forthcoming book [Joyal-Tierney, 2011]. We shall give here a new proof, which we think is fairly elementary — in particular it uses nothing more than the fact that the diagonal functor d sends level-wise weak homotopy equivalences to weak homotopy equivalences.

In the second part of the paper we present a simple construction of Kan's simplicial loop group functor as an application of Theorem 1.1. Recall that in [Kan, 1958], Kan defined a functor $G\colon \mathbf{S} \to s\mathbf{Gp}$ which is left adjoint to the classifying complex functor $\overline{W}\colon s\mathbf{Gp} \to \mathbf{S}$ of Eilenberg and Mac Lane [Eilenberg-Mac Lane, 1953]. He was able to show that, when X is reduced (i.e. when X is a simplicial set with only one vertex), the principal GX bundle X_η on X induced by the unit map $\eta\colon X \to \overline{W}GX$ has weakly contractible total space. Kan's proof of this last fact involves showing firstly that X_η is simply connected, and secondly that X_η is acyclic in the sense that it has vanishing reduced homology in all degrees.

We will show that both the construction of the functor G and the proof that the unit map is a weak homotopy equivalence can be greatly illuminated and simplified by considering a factorization (first noticed by Duskin) of $\overline{W}$ involving the functor T. In fact we hasten to point out that this last section of the paper makes no great claim to originality, we find it hard to believe that some of the results of this section were not known to Duskin, although we cannot find any evidence for this in his published papers. We also point out that in their forthcoming book [Joyal-Tierney, 2011] and their paper [Joyal-Tierney, 1996] Joyal and Tierney prove more general statements in the context of simplicially enriched groupoids. Using Duskin's factorization we will give a simple formula for the left adjoint to $\overline{W}$ (see Proposition 5.3). In Theorem 5.8 we will apply this formula to give a simple and direct proof of Kan's theorem that the unit map of the adjunction $G \dashv \overline{W}$ is a weak homotopy equivalence whenever X is reduced. To the best of our knowledge this proof is new (we note that an essential ingredient for the proof is Theorem 1.1). We point out that in [Waldhausen, 1996] Waldhausen described another approach to the construction of G, nevertheless we feel our approach (which proceeds along different lines) is still of some interest.

2. The décalage comonad

We begin by recalling the definition and main properties of the décalage and total décalage functors of Illusie [Illusie, 1972].

2.1. The décalage or shift functor Let Δ_a denote the augmented simplex category, in other words the simplex category Δ together with the additional object $[-1]$, the empty set (the initial object of Δ_a). We will write $as\mathscr{C}$ for the category $[\Delta_a^{\mathrm{op}}, \mathscr{C}]$ of augmented simplicial objects in a category $\mathscr{C}$, which we will assume to be complete and cocomplete. Recall (see for example VII.5 of [Mac Lane, 1998]) that Δ_a is a monoidal category with unit $[-1]$ under the operation of ordinal sum, which operation we will denote by σ (following Joyal and Tierney). If $[m]$, $[n] \in \Delta_a$ then $\sigma([m],[n]) = [m+n+1]$, and the operation σ gives rise to a bifunctor $\sigma\colon \Delta_a \times \Delta_a \to \Delta_a$ which sends a morphism

$$(\alpha, \beta)\colon ([m],[n]) \to ([m'],[n'])$$

in $\Delta_a \times \Delta_a$ to the morphism $\sigma(\alpha,\beta)\colon [m+n+1] \to [m'+n'+1]$ in Δ_a defined by

$$\sigma(\alpha,\beta)(i) = \begin{cases} \alpha(i) & \text{if } 0 \leq i \leq m \\ \beta(i-m-1)+m'+1 & \text{if } m+1 \leq i \leq m+n+1. \end{cases}$$

(Δ_a, σ) is not a symmetric monoidal category — while $\sigma([m],[n]) = \sigma([n],[m])$, it need not be the case that $\sigma(\alpha,\beta) = \sigma(\beta,\alpha)$. The monoidal structure on Δ_a allows us to define a functor $\sigma(-,[0])\colon \Delta_a \to \Delta$ which sends $[n] \in \Delta_a$ to $\sigma([n],[0]) = [n+1]$ in Δ. We have the following definition which we believe is originally due to Illusie.

2.2. Definition. [Illusie, 1972] *Define* $\mathrm{Dec}_0\colon s\mathscr{C} \to as\mathscr{C}$ *to be the functor given by restriction along* $\sigma(-,[0])\colon \Delta_a \to \Delta$*, so that if* X *is a simplicial object in* $\mathscr{C}$ *then* $\mathrm{Dec}_0 X$ *is the augmented simplicial object given by*

$$\mathrm{Dec}_0 X([n]) = X([n+1]),$$

whose face maps $d_i\colon (\mathrm{Dec}_0 X)_n \to (\mathrm{Dec}_0 X)_{n-1}$ *and degeneracy maps* $s_i\colon (\mathrm{Dec}_0 X)_n \to (\mathrm{Dec}_0 X)_{n+1}$ *for* $i = 0, 1, \ldots, n$ *are given by* $d_i\colon X_{n+1} \to X_n$ *and* $s_i\colon X_{n+1} \to X_{n+2}$ *respectively. The augmentation* $(\mathrm{Dec}_0 X)_0 \to X_0$ *is given by* $d_0\colon X_1 \to X_0$.

$\mathrm{Dec}_0 X$ is obtained from X by forgetting the top face and degeneracy map at each level and re-indexing by shifting degrees up by one. Thus the augmented simplicial object $\mathrm{Dec}_0 X$ can be pictured as

$$
\begin{tikzcd}
X_0 & X_1 \arrow[l, "d_0"'] & X_2 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"'] & X_3 \arrow[l, shift right=4, "d_0"'] \arrow[l, shift right=2, "d_1"'] \arrow[l, "d_2"'] & \cdots
\end{tikzcd}
$$

with degeneracies $s_0\colon X_1 \to X_2$ and $s_0, s_1\colon X_2 \to X_3$.

Note that the simplicial identity $d_0 d_1 = d_0 d_0$ shows that $d_0\colon X_1 \to X_0$ is an augmentation.

There is an analogous functor $\mathrm{Dec}^0\colon s\mathscr{C} \to as\mathscr{C}$ given by restriction along the functor $\sigma([0], -)\colon \Delta_a \to \Delta$ — thus Dec^0 is the functor which forgets the bottom face and degeneracy map at each level. The functors Dec_0 and Dec^0 are usually called the *décalage* or *shifting* functors. More generally we can define functors $\mathrm{Dec}_n\colon s\mathscr{C} \to as\mathscr{C}$ and $\mathrm{Dec}^n\colon s\mathscr{C} \to as\mathscr{C}$ induced by restriction along $\sigma(-, [n])\colon \Delta_a \to \Delta$ and $\sigma([n], -)\colon \Delta_a \to \Delta$ respectively.

The relation between $\mathrm{Dec}_n X$ and $\mathrm{Dec}^n X$ can be easily understood through the device of the opposite simplicial object. Let $\tau\colon \Delta \to \Delta$ denote the automorphism of Δ which reverses the order of each ordinal $[n]$, or equivalently sends the category $[n]$ to its opposite category. Note that $\tau(\sigma([m], [n])) = \sigma(\tau([n]), \tau([m]))$ for any $[m], [n] \in \Delta$. If X is a simplicial object then we write X^o for the simplicial object obtained by precomposing X with the functor τ^{op}. The simplicial object X^o is called the *opposite* simplicial object of X in [Joyal, 2008]. Note that $(\mathrm{Dec}_0 X)^o = \mathrm{Dec}^0(X^o)$ by the following calculation:

$$(\mathrm{Dec}_0 X)^o([n]) = \mathrm{Dec}_0 X(\tau([n])) = X(\sigma([0], \tau([n]))) = X(\sigma(\tau([n]), [0])),$$

since $\tau([0]) = [0]$. It follows that $(\mathrm{Dec}_n X)^o = \mathrm{Dec}^n(X^o)$ for any $n \geq 0$.

There are canonical comonads underlying the functors Dec_0 and Dec^0, when these functors are thought of as endofunctors on $s\mathscr{C}$ by forgetting augmentations. As is well known, $[0]$ determines a monoid in Δ whose multiplication is given by the canonical map $[1] \to [0]$. This monoid is universal in a certain precise sense (see Proposition 5.1 in Chapter VII of [Mac Lane, 1998]).

The monoid $[0]$ determines a corresponding comonoid in Δ^{op} which in turn induces by composition the two comonads Dec_0 and Dec^0 in $s\mathscr{C}$. The counit of the comonad Dec_0 is induced by the natural transformation $[n] \to \sigma([0], [n])$ and hence is given on a simplicial object X by the simplicial map $\mathrm{Dec}_0 X \to X$ which in degree n is the last face map $d_{n+1}\colon X_{n+1} \to X_n$.

Likewise, the counit of the comonad Dec^0 is induced by the natural transformation $[n] \to \sigma([n], [0])$ and hence is given on a simplicial object X by the simplicial map $\mathrm{Dec}^0 X \to X$ which in degree n is the first face map $d_0\colon X_{n+1} \to X_n$.

When Dec_0 and Dec^0 are regarded as endofunctors on $s\mathscr{C}$, we see that the functors Dec_n and Dec^n (also thought of as endofunctors on $s\mathscr{C}$) are given by $\mathrm{Dec}_n = (\mathrm{Dec}_0)^{n+1}$ and $\mathrm{Dec}^n = (\mathrm{Dec}^0)^{n+1}$ respectively.

2.3. Contractibility of the décalage functor It is an important fact that $\mathrm{Dec}_0 X$ and $\mathrm{Dec}^0 X$ are not just augmented simplicial objects, they are actually contractible augmented simplicial objects in the following sense.

Recall that the augmentation map $\epsilon\colon X \to X_{-1}$ of an augmented simplicial object X is a deformation retraction if there exists a simplicial map $s\colon X_{-1} \to X$ (with X_{-1} regarded as a constant simplicial object) which is a section of the projection ϵ and is such that $s\epsilon$ is homotopic to the identity map on X.

A sufficient condition for $s\epsilon$ to be homotopic to the identity map on X is that there exist for each $n \geq -1$, maps $s_{n+1}\colon X_n \to X_{n+1}$ with $s_0 = s$, which act as 'extra degeneracies on the right' in the sense that the following identities hold:

$$d_i s_n = s_{n-1} d_i \text{ for } 0 \leq i < n, \tag{1a}$$
$$d_n s_n = \mathrm{id}, \tag{1b}$$
$$s_i s_n = s_{n+1} s_i \text{ for } 0 \leq i \leq n, \tag{1c}$$

The following definition is standard.

2.4. Definition. *Let $\epsilon\colon X \to X_{-1}$ be an augmented simplicial object in $\mathscr{C}$. By a* contraction *of X we will mean the data of the section $s\colon X_{-1} \to X$ together with the extra degeneracies s_{n+1} as described above. We will say that X is* contractible *if it has such a contraction.*

A map of contractible augmented simplicial objects is a map of the underlying augmented simplicial objects which preserves the corresponding sections s and the extra degeneracies (as in [Duskin, 1975] we will sometimes say that such a map is *coherent*). We will write $a_c s\mathscr{C}$ for the category of contractible augmented simplicial objects and coherent maps.

Given the data of such a collection of maps s_{n+1} as above, we define maps $h_i\colon X_n \to X_{n+1}$ by the formula
$$h_i = s_0^{n-i} s_{n+1} d_0^{n-i}.$$
It is easy to check that the maps h_i satisfy the conditions (i)–(iii) in Definition 5.1 of [May, 1967]. The h_i then piece together to define a homotopy $h\colon X \otimes \Delta[1] \to X$ from $s\epsilon$ to the identity on X, analogous to Proposition 6.2 in [May, 1967]. Here, if K is a simplicial set, $X \otimes K$ denotes the tensor for the usual structure of $s\mathscr{C}$ as a simplicially enriched category, so that $X \otimes K$ has n-simplices given by

$$(X \otimes K)_n = \coprod_{k \in K_n} X_n. \tag{2}$$

In degree n, the map $h\colon X \otimes \Delta[1] \to X$ is given by $d_{i+1}h_i\colon (X_n)_\alpha \to X_n$ on the summand $(X_n)_\alpha$ of $(X \otimes \Delta[1])_n$ corresponding to the map $\alpha\colon [n] \to [1]$ determined by $\alpha^{-1}(0) = [i]$. We summarize this discussion in the following lemma.

2.5. Lemma. *Let $\epsilon\colon X \to X_{-1}$ be a contractible augmented simplicial object in $\mathscr{C}$. Then there is a homotopy $h\colon X \otimes \Delta[1] \to X$ in $s\mathscr{C}$ between $s\epsilon$ and 1_X.*

Clearly, the degeneracy $s_{n+1}\colon X_{n+1} \to X_{n+2}$ for $n \geq 0$ equips $\mathrm{Dec}_0 X$ with an extra degeneracy in the above sense. Therefore we have the following well known result.

2.6. Lemma. *For any simplicial object X in $\mathscr{C}$, the augmentation $d_0\colon \mathrm{Dec}_0 X \to X_0$ is a deformation retract. An analogous statement is true for $\mathrm{Dec}^0 X$.*

A prime example where simplicial objects with extra degeneracies appear is in the construction of simplicial comonadic resolutions. Let (L, δ, ϵ) be a comonad on a category $\mathscr{C}$, where $\epsilon\colon L \to 1$ and $\delta\colon L \to L^2$ denote the counit and the comultiplication of the comonad respectively. If X is an object of $\mathscr{C}$ then, as is well known, L determines an augmented simplicial object $L_* X$ with $L_n X = L^{n+1} X$ for $n \geq -1$, and whose face and degeneracy maps are defined by

$$d_i = L^i \epsilon L^{n-i},\ s_i = L^i \delta L^{n-i-1}$$

respectively. Let X be an object of $\mathscr{C}$ and suppose that $\sigma\colon X \to LX$ is a section of the counit $\epsilon\colon LX \to X$. Define a sequence of maps $s_{n+1}\colon L_n X \to L_{n+1} X$, $n \geq -1$ by

$$s_{n+1} = L^{n+1}\sigma\colon L^{n+1} X \to L^{n+2} X. \tag{3}$$

The following proposition gives a necessary and sufficient condition for this sequence of maps to define a contraction of the augmented simplicial object $L_* X$ (I am indebted to the referee for informing me of both this proposition and its proof, correcting an incomplete discussion in an earlier version of this paper).

2.7. Proposition. *Let $\sigma\colon X \to LX$ be a section of the counit $\epsilon\colon LX \to X$. Then the sequence of maps* (3) *is a contraction of the augmented simplicial object $L_* X$ if and only if (X, σ) is a coalgebra for the comonad L.*

Proof. The sequence of maps $s_{n+1}\colon L_n X \to L_{n+1} X$ above is a contraction of the augmented simplicial object $L_* X$ if and only if the equations (1) are satisfied. Of these equations, (1a) and (1b) follow from the naturality of ϵ, while the equation $s_i s_n = s_{n+1} s_i$ of (1c) follows from the naturality of δ for $0 \leq i < n$. When $i = n$, the equation (1c) amounts to the commutativity of the diagram

$$\begin{tikzcd} L^n X \arrow[r, "L^n\sigma"] \arrow[d, "L^n\sigma"'] & L^{n+1} X \arrow[d, "L^{n+1}\sigma"] \\ L^{n+1} X \arrow[r, "L^n\delta"'] & L^{n+2} X, \end{tikzcd}$$

which is assured by the commutativity of this same diagram in the case where $n = 0$, i.e. by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & LX \arrow[d, "L\sigma"] \\ LX \arrow[r, "\delta X"'] & L^2X. \end{tikzcd}$$

The commutativity of this diagram is in turn exactly the condition that σ is coassociative, in other words that (X, σ) is a coalgebra for (L, ϵ, δ). Thus the condition that (X, σ) is a coalgebra is both a necessary and a sufficient one, and the proposition follows. ∎

Lemma 2.5 immediately yields the following corollary.

2.8. Corollary. *Suppose that (X, σ) is a coalgebra for the comonad L. Then there is a homotopy $h\colon L_*X \otimes \Delta[1] \to L_*X$ in $s\mathscr{C}$ between $\sigma\epsilon$ and the identity on L_*X.*

3. The total décalage and the total simplicial set functors

In this section we will recall some of the main properties of Illusie's total décalage functor Dec [Illusie, 1972] and its right adjoint, the Artin-Mazur total simplicial set functor [Artin-Mazur, 1966]. For more details the reader should refer to the excellent discussion of these functors and their properties in the papers [Cegarra-Remedios, 2005, Cegarra-Remedios, 2007]. In this section we will mainly be interested in the case where $\mathscr{C} = \mathbf{Set}$. We begin therefore by explaining our notations and conventions for bisimplicial sets (which follows closely the presentation in [Joyal-Tierney, 2007]).

We write $\mathbf{SS}$ for the category of bisimplicial sets; if $X \in \mathbf{SS}$ is a bisimplicial set then we will say that $X_{m,n} = X([m],[n])$ has horizontal degree m and vertical degree n.We say a *simplicial space* is a simplicial object in $\mathbf{S}$. There are two ways in which we can regard a bisimplicial set X as a simplicial space. On the one hand, we can define for every $m \geq 0$ the simplicial set X_{m*} whose set of n-simplices is $(X_{m*})_n := X_{m,n}$. On the other hand we can define for every $n \geq 0$ the simplicial set X_{*n} whose set of m-simplices is $(X_{*n})_m := X_{m,n}$. Thus we may regard X as a horizontal simplicial object in $\mathbf{S}$ whose columns are the simplicial sets X_{m*}, or we may regard X as a vertical simplicial object in $\mathbf{S}$ whose rows are the simplicial sets X_{*n}. With the conventions above understood, we may sometimes use a shorthand for the columns X_{m*} by putting $X_m := X_{m*}$.

Each of these two ways of viewing a bisimplicial set as a simplicial space leads to a simplicial enrichment of $\mathbf{SS}$, using the canonical simplicial enrichment of $s\mathbf{S}$ mentioned earlier. If we view bisimplicial sets as horizontal simplicial objects in $\mathbf{S}$, then $\mathbf{SS} = s\mathbf{S}$ is equipped with the structure of a simplicial enriched category for which the tensor $X \otimes_1 K$, for X a bisimplicial set and $K \in \mathbf{S}$, has as its columns the simplicial sets given by (see (2))

$$(X \otimes_1 K)_m = (X \otimes_1 K)_{m*} = \coprod_{k \in K_m} X_m = X_m \times K_m,$$

so that the set of (m,n)-bisimplices of $X \otimes_1 K$ is $(X \otimes_1 K)_{m,n} = X_{m,n} \times K_m$. In other words,

$$X \otimes_1 K = X \times p_1^* K,$$

where $p_1 \colon \Delta \times \Delta \to \Delta$ denotes projection onto the first factor. The simplicial enrichment is then defined by the formula

$$\mathrm{Hom}_1(X, Y) = (p_1)_*(Y^X),$$

where $(p_1)_*$ denotes the right adjoint to $(p_1)^*$; observe that since $i_1 \colon \Delta \to \Delta \times \Delta$ defined by $i_1([n]) = ([n], [0])$ is right adjoint to p_1, it follows that $(p_1)_* \colon \mathbf{SS} \to \mathbf{S}$ is the functor which sends a bisimplicial set to the simplicial set which is its first row.

Similarly, if we view $X \in \mathbf{SS}$ as a vertical simplicial object, then the tensor $X \otimes_2 K$ is given by

$$X \otimes_2 K = X \times p_2^* K$$

where $p_2 \colon \Delta \times \Delta \to \Delta$ denotes projection onto the second factor. The simplicial enrichment is defined by the formula

$$\mathrm{Hom}_2(X, Y) = (p_2)_*(Y^X),$$

where $(p_2)_*$ denotes the right adjoint to p_2^*, i.e. the functor $(p_2)_* \colon \mathbf{SS} \to \mathbf{S}$ which sends a bisimplicial set to its first column.

We write $A \boxtimes B$ for the *external product* of simplicial sets A and B; this is the bisimplicial set with $(A \boxtimes B)_{m,n} = A_m \times B_n$. In particular we write $\Delta[m,n] := \Delta[m] \boxtimes \Delta[n]$ for the *prism* of shape (m,n). Observe that $p_1^* A = A \boxtimes 1$, $p_2^* B = 1 \boxtimes B$ and $p_1^* A \times p_2^* B = A \boxtimes B$.

Following Joyal we will say that a *row augmentation* of a bisimplicial set X by a simplicial set A is a map $X \to p_1^* A$ in $\mathbf{SS}$; thus a row augmentation is the same thing as a map of bisimplicial sets $X \to A \boxtimes 1$. Similarly a *column augmentation* of X by a simplicial set B is a map $X \to p_2^* B$ in $\mathbf{SS}$; thus a column augmentation is the same thing as a map of bisimplicial sets $X \to 1 \boxtimes B$. A *double augmentation* is a map of bisimplicial sets $X \to A \boxtimes B$.

The category $\mathbf{SS}$ has two intervals, the *horizontal interval* $\Delta[1,0]$ and the *vertical interval* $\Delta[0,1]$. Accordingly, there are two notions of homotopy in $\mathbf{SS}$: a *horizontal homotopy* is a map $X \times \Delta[1,0] \to Y$ and a *vertical homotopy* is a map $X \times \Delta[0,1] \to Y$. Note that a vertical homotopy $X \times \Delta[0,1] \to Y$ induces an ordinary homotopy on the columns of X and Y, while a horizontal homotopy $X \times \Delta[1,0] \to Y$ induces an ordinary homotopy on the rows of X and Y.

With these conventions understood we can describe Illusie's total décalage functor [Illusie, 1972]. The simplicial comonadic resolution of Dec_0 gives rise to a functor $\mathrm{Dec} \colon \mathbf{S} \to s\mathbf{S}$ which sends a simplicial set X to the simplicial space $\mathrm{Dec}X$ which in degree n is the (horizontal) simplicial set

$$\mathrm{Dec}_n X = (\mathrm{Dec}_0)^{n+1} X.$$

Here we are thinking of $\mathrm{Dec}X$ as a vertical simplicial object in $\mathbf{S}$ with horizontal simplicial sets. The set of (m,n)-bisimplices of the bisimplicial set $\mathrm{Dec}X$ is $(\mathrm{Dec}X)_{m,n} =$

X_{m+n+1}. The horizontal and vertical face operators $d_i^h\colon (\mathrm{Dec}\, X)_{m+1,n} \to (\mathrm{Dec}\, X)_{m,n}$ and $d_i^v\colon (\mathrm{Dec}\, X)_{m,n+1} \to (\mathrm{Dec}\, X)_{m,n}$ are given by $d_i^h = d_i\colon X_{m+n+2} \to X_{m+n+1}$ and $d_i^v = d_{m+i+1}\colon X_{m+n+2} \to X_{m+n+1}$ respectively. There are similar formulas for the horizontal and vertical degeneracy operators. Note that if we regard the bisimplicial set $\mathrm{Dec}X$ as a horizontal simplicial object with vertical simplicial sets then $\mathrm{Dec}X$ is the simplicial comonadic resolution of X by Dec^0. The following Lemma is straightforward.

3.1. LEMMA. *The functor* $\mathrm{Dec}\colon \mathbf{S} \to \mathbf{SS}$ *is given by restriction along the ordinal sum map* $\sigma\colon \Delta \times \Delta \to \Delta$*, so that*

$$\mathrm{Dec}\, X([m],[n]) = X(\sigma([m],[n])) = X_{n+m+1}$$

for $X \in \mathbf{S}$.

If X is a simplicial set then the bisimplicial set $\mathrm{Dec}X$ is called the *total décalage* of X. Note that in fact $\mathrm{Dec}X$ comes equipped with a natural double augmentation in the above sense. To see this note that the n-th row of $\mathrm{Dec}X$ is the simplicial set $\mathrm{Dec}_n X$ which has the augmentation $d_0\colon X_{1+n} \to X_n$. These augmentations assemble together to define the column augmentation $\epsilon_c\colon \mathrm{Dec}X \to 1 \boxtimes X$ of the bisimplicial set X. Dually, the m-th column of $\mathrm{Dec}X$ is the simplicial set $\mathrm{Dec}^m X$ which has the augmentation $d_{m+1}\colon X_{m+1} \to X_m$; these augmentations assemble together to define the row augmentation $\epsilon_r\colon \mathrm{Dec}X \to X \boxtimes 1$ of the bisimplicial set $\mathrm{Dec}X$.

I am indebted to the referee for the conceptual proof of the following lemma, and for a suggestion on how to streamline the statement of the lemma.

3.2. LEMMA. *Let* X *be a simplicial set. Then the map* $\epsilon_r\colon \mathrm{Dec}X \to X \boxtimes 1$ *is a column-wise homotopy equivalence and the map* $\epsilon_c\colon \mathrm{Dec}X \to 1 \boxtimes X$ *is a row-wise homotopy equivalence. Moreover,* ϵ_r *is a vertical homotopy equivalence and* ϵ_c *is a horizontal homotopy equivalence in the case where* $X = \Delta[n]$.

PROOF. Observe that the n-th row of ϵ_c is the augmentation $d_0\colon \mathrm{Dec}_n X \to X_n$ and that $\mathrm{Dec}_n X = \mathrm{Dec}_0(\mathrm{Dec}_0)^n X$; therefore Lemma 2.5 applies to show that $d_0\colon \mathrm{Dec}_n X \to X_n$ is a homotopy equivalence. The dual statement for ϵ_r follows immediately.

We now prove the more refined statement in the case where $X = \Delta[n]$: let us show that $\epsilon_r\colon \mathrm{Dec}X \to X \boxtimes 1$ is a vertical homotopy equivalence in this case. As noted earlier, the functor $L = \mathrm{Dec}_0\colon \mathbf{S} \to \mathbf{S}$ has the structure of a comonad and the associated augmented simplicial object $L_*\Delta[n]$ is the augmented simplicial object $\epsilon_r\colon \mathrm{Dec}\Delta[n] \to \Delta[n] \boxtimes 1$. By Proposition 2.7, it suffices to show that $\Delta[n]$ has the structure of a coalgebra over the comonad L, it will then follow from Corollary 2.8 that the augmentation $\epsilon_r\colon L_*\Delta[n] \to \Delta[n]$ is a vertical homotopy equivalence. But the functor $L = \mathrm{Dec}_0$ has a left adjoint $S\colon \mathbf{S} \to \mathbf{S}$ which is the left Kan extension of the functor $\sigma(-,[0])\colon \Delta \to \Delta$. In particular, we have $S\Delta[n] = \Delta[n+1]$ for every $n \geq 0$. The functor S has the structure of a monad, since Dec_0 has the structure of a comonad. Therefore it suffices to show that $\Delta[n]$ has the structure of an algebra over the monad S. The unit of the monad S is the inclusion $d_{n+1}\colon \Delta[n] \to S\Delta[n]$ and its multiplication is the map $s_{n+1}\colon S^2\Delta[n] \to S\Delta[n]$; it is then easy to verify that $s_n\colon S\Delta[n] \to \Delta[n]$ is a S-algebra structure on $\Delta[n]$. The proof that $\epsilon_c\colon L_*\Delta[n] \to \Delta[n]$ is a horizontal homotopy equivalence is similar. ∎

From the description of Dec in Lemma 3.1 above it is clear that Dec has both a left and right adjoint. The left adjoint of Dec is related to the notion of the join of simplicial sets. The right adjoint to Dec is denoted $T\colon \mathbf{SS} \to \mathbf{S}$, it was introduced in [Artin-Mazur, 1966] where it was called the *total simplicial set functor*. It is also known as the *Artin-Mazur codiagonal*. It has the following explicit description: if X is a bisimplicial set then the set $(TX)_n$ of n-simplices of the simplicial set TX is given by the equalizer of the diagram

$$(TX)_n \to \prod_{i=0}^{n} X_{i,n-i} \rightrightarrows \prod_{i=0}^{n-1} X_{i,n-i-1} \tag{4}$$

where the components of the two maps are defined by the composites

$$\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_i} X_{i,n-i} \xrightarrow{d_0^v} X_{i,n-i-1}$$

and

$$\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_{i+1}} X_{i+1,n-i-1} \xrightarrow{d_{i+1}^h} X_{i,n-i-1}.$$

The face maps $d_i\colon (TX)_n \to (TX)_{n-1}$ are given by

$$d_i = (d_i^v p_0, d_{i-1}^v p_1, \dots, d_1^v p_{i-1}, d_i^h p_{i+1}, d_i^h p_{i+2}, \dots, d_i^h p_n)$$

while the degeneracy maps $s_i\colon (TX)_n \to (TX)_{n+1}$ are given by

$$s_i = (s_i^v p_0, s_{i-1}^v p_1, \dots, s_0^v p_i, s_i^h p_{i+1}, \dots, s_i^h p_n).$$

The unit map $\eta\colon X \to T\mathrm{Dec}\, X$ of the adjunction $\mathrm{Dec} \dashv T$ is given by the map

$$x \mapsto (s_0(x), s_1(x), \dots, s_n(x)) \tag{5}$$

in degree n (see [Cegarra-Remedios, 2005]). In general it is rather difficult to give a simple description of the simplicial set TX for an arbitrary bisimplicial set X. When X is constant however, we have the following well-known result.

3.3. Lemma. *Let X be a simplicial set. Then there are isomorphisms $Tp_1^*X = Tp_2^*X = X$, natural in X.*

Proof. Observe that the functor Tp_1^* is right adjoint to the functor $\pi_0\mathrm{Dec}$. Lemma 3.2 implies that the functor $\pi_0\mathrm{Dec}$ is the identity on $\mathbf{S}$, from which it follows that there is an isomorphism $Tp_1^*X = X$, natural in X. The other statement is proven in an analogous fashion. ∎

Since $T\Delta[1,0] = Tp_1^*\Delta[1] = \Delta[1]$ we immediately obtain the following result (which was also pointed out to me by the referee).

3.4. PROPOSITION. *The functor* $T\colon \mathbf{SS} \to \mathbf{S}$ *takes a horizontal (respectively vertical) homotopy of bisimplicial maps to an ordinary homotopy of simplicial maps.*

4. The generalized Eilenberg-Zilber theorem for simplicial sets

Our goal in this section is to present an elementary proof of Theorem 1.1. Recall that this theorem states that there is a weak equivalence

$$dX \to TX \tag{6}$$

of simplicial sets, natural in X. As mentioned earlier, the proof of Theorem 1.1 in [Cegarra-Remedios, 2005] is rather lengthy, and so it is of interest to have a simpler approach. We will describe here another proof, which we think is fairly elementary (as stated in the Introduction, the forthcoming book [Joyal-Tierney, 2011] of Joyal and Tierney contains another proof, which proceeds along different lines).

We begin by describing the map (6). This map is induced by the map of cosimplicial bisimplicial sets

$$(\epsilon_r, \epsilon_c)\colon \mathrm{Dec}\Delta \to (\Delta \boxtimes \Delta)\delta, \tag{7}$$

i.e. the map which in degree n is the double augmentation $(\epsilon_r, \epsilon_c)\colon \mathrm{Dec}\Delta[n] \to \Delta[n]\boxtimes\Delta[n]$. Since $dX_n = \mathbf{SS}(\Delta[n,n], X) = \mathbf{SS}(\Delta[n] \boxtimes \Delta[n], X)$ and $TX_n = \mathbf{SS}(\mathrm{Dec}\Delta[n], X)$ we see that (7) does give rise to a map $dX \to TX$ in this way. Note that it is possible to describe the map $dX \to TX$ much more explicitly at the level of simplices (see [Cegarra-Remedios, 2005]) but we will not need this.

The proof of Theorem 1.1 that we shall give essentially boils down to the well known fact that the diagonal functor $d\colon \mathbf{SS} \to \mathbf{S}$ sends level-wise weak homotopy equivalences of bisimplicial sets to weak homotopy equivalences of simplicial sets. In other words, if $f\colon X \to Y$ is a map in $\mathbf{SS}$ such that the map $f_{*n}\colon X_{*n} \to Y_{*n}$ on n-th rows is a weak homotopy equivalence for all $n \geq 0$, then $df\colon dX \to dY$ is also a weak homotopy equivalence. Alternatively, if the map $f_{m*}\colon X_{m*} \to Y_{m*}$ on the m-th columns is a weak homotopy equivalence for all $m \geq 0$, then $df\colon dX \to dY$ is a weak homotopy equivalence.

Recall that d has a right adjoint $d_*\colon \mathbf{S} \to \mathbf{SS}$ (see for instance [Goerss-Jardine, 1999] page 222) defined by the formula

$$(d_*X)_{m,n} = \mathbf{S}(\Delta[m] \times \Delta[n], X). \tag{8}$$

Using the fact that the diagonal d sends level-wise weak equivalences to weak equivalences one can prove (see for instance [Moerdijk, 1989]) that the counit $\epsilon\colon dd_*K \to K$ of this adjunction is a weak homotopy equivalence for any simplicial set K, and so in particular $dd_*TX \to TX$ is a weak homotopy equivalence for any bisimplicial set X. Therefore, since we can factor (6) as

$$dX \to dd_*TX \to TX,$$

we see that to prove Theorem 1.1 it suffices to prove the following proposition.

4.1. Proposition. *The map $dX \to dd_*TX$ is a weak homotopy equivalence for any bisimplicial set X.*

I am very grateful to the referee for several helpful suggestions which have helped to simplify the following proof.

Proof. First observe that, by definition, for any bisimplicial set X we have

$$\begin{aligned}(d_*TX)_{m,n} &= \mathbf{SS}(\mathrm{Dec}\, d\,\Delta[m,n], X)\\ &= \mathbf{SS}(\mathrm{Dec}\Delta[m] \times \mathrm{Dec}\Delta[n], X).\end{aligned}$$

Using the observation that the map (7) factors as

$$\mathrm{Dec}\Delta \to \mathrm{Dec}(\Delta \times \Delta)\delta \to (\Delta \boxtimes \Delta)\delta,$$

where the map $\Delta \to (\Delta \times \Delta)\delta$ is the canonical map inducing the counit $dd_* \to 1$ of the adjunction $d \dashv d_*$, we see that the natural map $X \to d_*TX$ is induced by the map of bicosimplicial bisimplicial sets

$$\mathrm{Dec}\,\Delta \times \mathrm{Dec}\,\Delta \xrightarrow{1\times\epsilon_c} \mathrm{Dec}\,\Delta \times p_2^*\Delta \xrightarrow{\epsilon_r\times 1} \Delta \boxtimes \Delta.$$

It follows therefore that the natural map $X \to d_*TX$ factorizes as

$$X \xrightarrow{\alpha} RX \xrightarrow{\beta} d_*TX,$$

where $R\colon \mathbf{SS} \to \mathbf{SS}$ denotes the functor which sends a bisimplicial set X to the bisimplicial set

$$RX = \mathbf{SS}(\mathrm{Dec}\Delta \times p_2^*\Delta, X).$$

We will show that the map α is a column-wise homotopy equivalence and that the map β is a row-wise homotopy equivalence; it will then follow that $X \to d_*TX$ is a diagonal weak homotopy equivalence, as we claim.

We begin by examining the rows of the bisimplicial set RX: for any $m, n \geq 0$ we have

$$\begin{aligned}(RX)_{m,n} &= \mathbf{SS}(\mathrm{Dec}\Delta[m] \times p_2^*\Delta[n], X)\\ &= \mathbf{SS}(\mathrm{Dec}\Delta[m], X^{p_2^*\Delta[n]})\\ &= \mathbf{S}(\Delta[m], T(X^{p_2^*\Delta[n]})),\end{aligned}$$

from which it follows that the n-th row of RX is the simplicial set $T(X^{p_2^*\Delta[n]})$. Likewise we have

$$\begin{aligned}(d_*TX)_{m,n} &= \mathbf{SS}(\mathrm{Dec}\Delta[m] \times \mathrm{Dec}\Delta[n], X)\\ &= \mathbf{SS}(\mathrm{Dec}\Delta[m], X^{\mathrm{Dec}\Delta[n]})\\ &= \mathbf{S}(\Delta[m], T(X^{\mathrm{Dec}\Delta[n]})),\end{aligned}$$

from which it follows that the n-th row of d_*TX is the simplicial set $T(X^{\mathrm{Dec}\Delta[n]})$. Moreover the n-th row of the map β is plainly the map $T(X^{\epsilon_c})\colon T(X^{p_2^*\Delta[n]}) \to T(X^{\mathrm{Dec}\Delta[n]})$, where $\epsilon_c\colon \mathrm{Dec}\Delta[n] \to p_2^*\Delta[n]$ is the column augmentation of $\mathrm{Dec}\Delta[n]$. We have seen in Lemma 3.2 that the map ϵ_c is a horizontal homotopy equivalence. It follows, since **SS** is cartesian closed, that X^{ϵ_c} is a horizontal homotopy equivalence and hence that $T(X^{\epsilon_c})$ is a homotopy equivalence by Proposition 3.4.

Next we examine the columns of the bisimplicial set RX: for any $m, n \geq 0$ we have

$$
\begin{aligned}
(RX)_{m,n} &= \mathbf{SS}(\mathrm{Dec}\Delta[m] \times p_2^*\Delta[n], X) \\
&= \mathbf{SS}(p_2^*\Delta[n], X^{\mathrm{Dec}\Delta[m]}) \\
&= \mathbf{S}(\Delta[n], (p_2)_* X^{\mathrm{Dec}\Delta[m]}),
\end{aligned}
$$

from which it follows that m-th column of RX is the simplicial set $(p_2)_*X^{\mathrm{Dec}\Delta[m]}$. Writing $\Delta[m,n] = p_1^*\Delta[m] \times p_2^*\Delta[n]$ and using adjointness shows that the m-th column of X is the simplicial set $(p_2)_*X^{p_1^*\Delta[m]}$ and that the m-th column of the map α is the map

$$
(p_2)_*X^{\epsilon_r}\colon (p_2)_*X^{p_1^*\Delta[m]} \to (p_2)_*X^{\mathrm{Dec}\Delta[m]},
$$

where $\epsilon_r\colon \mathrm{Dec}\Delta[m] \to p_1^*\Delta[m]$ denotes the row augmentation of $\mathrm{Dec}\Delta[m]$. Lemma 3.2 shows that ϵ_r is a vertical homotopy equivalence; it then follows that X^{ϵ_r} is a vertical homotopy equivalence and hence that $(p_2)_*X^{\epsilon_r}$ is a homotopy equivalence (recall that $(p_2)_*\colon \mathbf{SS} \to \mathbf{S}$ is the functor which sends a bisimplicial set to its first column). ■

5. Kan's simplicial loop group construction revisited

The classifying complex $\overline{W}G$ of a simplicial group G was introduced in [Eilenberg-Mac Lane, 1953] (see Section 17 of that paper). We recall the definition.

5.1. Definition. [Eilenberg-Mac Lane, 1953] *Let G be a simplicial group. Then $\overline{W}G$ is the simplicial set with a single vertex, and whose set of n-simplices, $n \geq 1$, is given by*

$$
(\overline{W}G)_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0.
$$

The face and degeneracy maps of $\overline{W}G$ are given by the following formulas:

$$
d_i(g_{n-1}, \ldots, g_0) = \begin{cases} (g_{n-2}, \ldots, g_0) & \text{if } i = 0, \\ (d_i(g_{n-1}), \ldots, d_1(g_{n-i+1}), g_{n-i-1}d_0(g_{n-i}), g_{n-i-2}, \ldots, g_0) & \text{if } 1 \leq i \leq n \end{cases}
$$

and

$$
s_i(g_{n-1}, \ldots, g_0) = \begin{cases} (1, g_{n-1}, \ldots, g_0) & \text{if } i = 0, \\ (s_{i-1}(g_{n-1}), \ldots, s_0(g_{n-i}), 1, g_{n-i-1}, \ldots, g_0) & \text{if } 1 \leq i \leq n. \end{cases}
$$

The motivation for the above formula for $\overline{W}G$ is perhaps not so clear. We will show that there is a very natural 'explanation' for the above formula in terms of the décalage functors. For this, we first need some background on principal twisted cartesian products.

Recall that a *principal twisted cartesian product* (PTCP) with structure group G consists of a simplicial set P (the total space) and a simplicial set M (the base space) together with a map $\pi\colon P \to M$ and an action of G on P which is principal in the sense that the diagram

$$\begin{tikzcd} P \times G \arrow[r] \arrow[d, "p_1"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & M \end{tikzcd}$$

is a pullback, where p_1 denotes projection onto the first factor, the top arrow is the action of G on P, and π denotes the projection to the base. Moreover, $\pi\colon P \to M$ is required to have a *pseudo-cross section* (on the left), i.e. a family of sections σ_n of the maps $\pi_n\colon P_n \to M_n$ for all $n \geq 0$ such that $\sigma_{n+1}s_i = s_i\sigma_n$ for all $0 \leq i \leq n$ and $d_i\sigma_n = \sigma_{n-1}d_i$ for all $0 < i \leq n$.

The simplicial set $\overline{W}G$ is a classifying space for PTCPs with structure group G in the sense that there is a universal PTCP WG with base space $\overline{W}G$ with the property that every PTCP P on M with structure group G is induced by pullback from $WG \to \overline{W}G$ along a map $M \to \overline{W}G$, the classifying map of P.

In [Duskin, 1975] Duskin explained how this classical notion of pseudo-cross section has a convenient reformulation in terms of Dec^0. In this reformulation, σ is required to be a section of the induced map $\mathrm{Dec}^0\pi\colon \mathrm{Dec}^0P \to \mathrm{Dec}^0M$ in the category $a_c\mathbf{S}$ of contractible augmented simplicial sets and coherent maps (see Section 2.3).

Since G acts principally on P, there is a canonical map of bisimplicial sets

$$\mathrm{cosk}_0 P \to NG,$$

where NG denotes the bisimplicial set which, when viewed as a (vertical) simplicial object in $\mathbf{S}$, has as its object of n-simplices the (horizontal) simplicial set NG_n, i.e. the nerve of the group G_n. Also here $\mathrm{cosk}_0 P$ denotes the 0-coskeleton (or Čech nerve) of P, viewed as an object in $\mathbf{S}/M$. Therefore, $\mathrm{cosk}_0 P$ has as its object of n-simplices the (horizontal) simplicial set $\check{C}(P_n)$ which is the Čech nerve of the map $\pi_n\colon P_n \to M_n$. In degree n the canonical map $\mathrm{cosk}_0 \to NG$ is just the canonical map $\check{C}(P_n) \to NG_n$ arising from the principal action of G_n on P_n.

One of the advantages of this reformulation of the notion of PTCP is that it allows for a very simple and conceptual description of the classifying map of P (we find it hard to believe that this description was not known to Duskin). We have a commutative diagram

$$\begin{tikzcd} \mathrm{Dec}^0 P \arrow[r] \arrow[d] & P \arrow[d] \\ \mathrm{Dec}^0 M \arrow[r] & M. \end{tikzcd}$$

Composing the pseudo-cross section $s\colon \mathrm{Dec}^0 M \to \mathrm{Dec}^0 P$ with the map $\mathrm{Dec}^0 P \to P$ gives rise to a map $\mathrm{Dec}^0 M \to P$ over M which extends canonically to a simplicial map

$$\mathrm{Dec}\, M \to \mathrm{cosk}_0 P \tag{9}$$

between simplicial objects in $\mathbf{S}/M$. Here $\mathrm{Dec}^0 M$ is thought of as the vertical simplicial set of 0-simplices of the bisimplicial set $\mathrm{Dec}\, M$. We can compose (9) with the canonical map $\mathrm{cosk}_0 P \to NG$ to obtain a map $\mathrm{Dec}\, M \to NG$. The adjoint of the map $\mathrm{Dec}\, M \to NG$ is a map

$$M \to TNG$$

which serves as a classifying map for P. One can go further and show that there is a canonical PTCP with base space TNG from which P arises via pullback along the above map. The next result shows that TNG is *precisely* the classifying complex $\overline{W}G$.

5.2. Lemma. [Duskin] *The classifying complex functor $\overline{W}$ factors as*

$$\overline{W} = TN,$$

so that $\overline{W}G = TNG$ for any simplicial group G.

This factorization of $\overline{W}$ is due as far as we know to Duskin, who observed that it persists when simplicial groups are replaced by simplicially enriched groupoids, i.e. the functor $\overline{W}\colon \mathbf{SGpd} \to \mathbf{S}$ introduced by Dwyer and Kan in [Dwyer-Kan, 1984] also factors as $\overline{W} = TN$ (this last observation also appears in the MSc thesis of Ehlers [Ehlers, 1991]).

Proof. This is an essentially straightforward computation, so we will just give a sketch of the details. To an n-simplex of $\overline{W}G$ consisting of a tuple

$$(g_{n-1}, g_{n-2}, \ldots, g_0)$$

as above, we associate the element $(x_0, x_1, \ldots, x_n)$ of TNG, where $x_0 = 1$ and

$$x_i = (d_0^{i-1}(g_{n-1}), d_0^{i-2}(g_{n-2}), \ldots, d_0(g_{n-i+1}), g_{n-i}) \in (NG_{n-i})_i$$

for $i \geq 1$. This sets up a bijection $(\overline{W}G)_n = (TNG)_n$ which respects face and degeneracy maps. ∎

It is well known that $\overline{W}G$ is weakly equivalent to the simplicial set dNG, obtained by applying the diagonal functor to the degree-wise nerve NG of the simplicial group G. Of course this can be seen as an instance of Theorem 1.1 in light of the identification $\overline{W}G = TNG$, but there are easier proofs, see for example [Jardine-Luo, 2006]. In fact, $\overline{W}G$ is simplicially homotopy equivalent to dNG, the point being that both $\overline{W}G$ and dNG are fibrant (a proof of the latter fact can be found in [Joyal-Tierney, 1996]). In [Thomas, 2008] it is shown via explicit calculation that the map $f\colon dNG \to \overline{W}G$ defined by

$$f(h_1, \ldots, h_n) = (d_0(h_1), \ldots, d_0^n(h_n))$$

for $h_i \in G_n$ exhibits dNG as a deformation retract of $\overline{W}G$. There is a further relationship between dNG and $\overline{W}G$ (see [Berger-Huebschmann, 1998]): after passing to geometric realizations there is an *isomorphism* of spaces $|\overline{W}G| = |dNG|$. It is not clear that this isomorphism is induced by a simplicial map however. It would be interesting to give a more conceptual proof of the isomorphism from [Berger-Huebschmann, 1998].

There are several advantages of the description of $\overline{W}$ in Lemma 5.2 over the traditional description. One such advantage of the present description is that it becomes manifestly clear that $\overline{W}$ has a left adjoint since both of the functors N and T do.

5.3. Proposition. *A left adjoint for the functor $\overline{W} = TN$ is given by the functor*

$$G = \pi_1 R\,\mathrm{Dec}\colon \mathbf{S} \to s\mathbf{Gp},$$

where $R\colon \mathbf{SS} \to s\mathbf{S}_0$ is the left adjoint of the inclusion $s\mathbf{S}_0 \subset \mathbf{SS}$. If X is a simplicial set, then the value of G on X is the simplicial group GX defined by

$$[n] \mapsto \pi_1(\mathrm{Dec}_n X/X_{n+1}).$$

Proof. Observe that the functor R is induced by the left adjoint of the inclusion $\mathbf{S}_0 \subset \mathbf{S}$, i.e. the functor which sends a simplicial set X to the reduced simplicial set $X/\mathrm{sk}_0 X$. To describe RX for X a bisimplicial set whose n-th row is X_{*n}, we let $\mathrm{sk}_0 X$ denote the bisimplicial set whose n-th row is $\mathrm{sk}_0 X_{*n}$, i.e. the constant simplicial set $[m] \mapsto X_{0,n}$. Then $RX = X/\mathrm{sk}_0 X$ so that the n-th row of RX is $(RX)_{*n} = X_{*n}/X_{0,n}$. The proposition then follows from the fact that $\mathrm{sk}_0\mathrm{Dec}_n X$ is the constant simplicial set X_{n+1}. ∎

Recall that a simplicial group G is said to be a *loop group* for a simplicial set X if there is a PTCP P on X with structure group G such that P is weakly contractible. In [Kan, 1958] Kan showed that the left adjoint $G\colon \mathbf{S} \to s\mathbf{Gp}$ of the classifying complex functor $\overline{W}$ had the property that $G(X)$ was a loop group for any reduced simplicial set X. We will shortly give a simplified proof of his theorem by exploiting the description of G given in Proposition 5.3 above. Before we do this however we need the following lemmas.

5.4. Lemma. *Suppose that X is a bisimplicial set whose first column is weakly contractible, i.e. the simplicial set $[n] \mapsto X_{0,n}$ is weakly contractible. Then $X \to RX$ is a column-wise weak equivalence.*

Proof. For every $m \geq 0$, the vertical simplicial set $(\mathrm{sk}_0 X)_m$ is weakly contractible and so $X_m \to X_m/(\mathrm{sk}_0 X)_m$ is a weak equivalence of vertical simplicial sets for every $m \geq 0$. ∎

5.5. Lemma. *Let X be a CW complex whose path components are all contractible. Then X/X^0 is a $K(\pi, 1)$, where X^0 denotes the set of vertices of X.*

Proof. X/X^0 can be written as a wedge

$$\bigvee_{\alpha \in \pi_0(X)} X_\alpha / X_\alpha^0,$$

where X_α denote the path components of X. Therefore without loss of generality we can assume that X is a path connected, pointed CW complex. We then have to show that X/X^0 is a $K(\pi, 1)$. Choose a strong deformation retraction of X onto a maximal tree T in the 1-skeleton X^1 of X (see for example I Theorem 5.9 of [Whitehead, 1978]). Then T/X^0 is a deformation retract of X/X^0 and so X/X^0 is a wedge of circles, from which the result follows. ∎

5.6. Corollary. *For any simplicial set X, $\mathrm{Dec}_n X/X_{n+1}$ has the weak homotopy type of a $K(\pi, 1)$.*

Proof. Since $\mathrm{Dec}_n X = \mathrm{Dec}_0\mathrm{Dec}_{n-1}X$, it is enough to prove this for $\mathrm{Dec}_0 X/X_1$. However this follows immediately from the Lemma since $\mathrm{Dec}_0 X$ deformation retracts onto X_0 (see Lemma 2.6). ∎

With a little extra effort one can use this corollary to construct an explicit isomorphism between GX and the simplicial group described by Kan in [Kan, 1958], however we will not do this here.

We can now give a simple proof of Kan's result from [Kan, 1958] that $X \to \overline{W}GX$ is a weak homotopy equivalence when X is reduced. We will need the following property of the total simplicial set functor T: as observed in [Cegarra-Remedios, 2005], since d sends level-wise weak homotopy equivalences of bisimplicial sets to weak homotopy equivalences of simplicial sets, Theorem 1.1 implies that this property is inherited by T. As an immediate consequence of this observation, Cegarra and Remedios prove the following:

5.7. Lemma. [Cegarra-Remedios, 2005] *For any simplicial set X, the unit map $X \to T\mathrm{Dec}\, X$ is a weak homotopy equivalence.*

We briefly review the proof of this result from [Cegarra-Remedios, 2005].

Proof. Cegarra and Remedios observe that the composite of the unit $X \to T\mathrm{Dec}X$ with the map $T\mathrm{Dec}X \to Tp_1^*X$ is the identity on X, in light of the identification $Tp_1^*X = X$ of Lemma 3.3. Since T sends level-wise weak homotopy equivalences to weak equivalences it follows that $T\mathrm{Dec}X \to X$ is a weak homotopy equivalence and hence the unit map is a weak homotopy equivalence. ∎

We are now ready to prove that GX is a loop group for X whenever X is reduced.

5.8. Theorem. [Kan, 1958] *Let X be a reduced simplicial set. Then the unit map*

$$\eta\colon X \to \overline{W}GX$$

is a weak homotopy equivalence. Hence GX is a loop group for X.

Proof. The units of the adjunctions $\mathrm{Dec} \dashv T$, $R \dashv U$, and $N_0 \dashv \pi_1$ give a factorization of η

$$X \to T\mathrm{Dec}X \to TR\,\mathrm{Dec}\,X \to TN\pi_1 R\,\mathrm{Dec}\,X$$

in **S**. The map $X \to T\mathrm{Dec}\, X$ is a weak homotopy equivalence by Lemma 5.7. The maps $T\mathrm{Dec}\, X \to TR\,\mathrm{Dec}\, X$ and $TR\,\mathrm{Dec}\, X \to T\pi_1 R\,\mathrm{Dec}\, X$ are induced by the maps

$$\mathrm{Dec}\, X \to R\,\mathrm{Dec}\, X \text{ and } R\,\mathrm{Dec}\, X \to N\pi_1 R\,\mathrm{Dec}\, X$$

in **SS**. We will show that both of these maps are level-wise weak homotopy equivalences. The first map is a level-wise weak homotopy equivalence by Lemma 5.4, since $\mathrm{sk}_0\mathrm{Dec}X = \mathrm{Dec}^0 X$ and X is reduced. Corollary 5.6 shows that $R\,\mathrm{Dec}_n X$ has the weak homotopy type of a $K(\pi, 1)$ and so the second map is also a level-wise weak homotopy equivalence. ∎

Acknowledgements I thank Tom Leinster for some useful conversations. I am grateful to Pouya Adrom, John Baez, Tim Porter, Birgit Richter and David Roberts for useful comments. I thank Jean-Louis Loday for the reference to Quillen's paper. I am especially grateful to Myles Tierney for informing me about his work with Joyal and for several very useful conversations; to Sebastian Thomas for pointing out an erroneous statement regarding the results of Dold and Puppe, and for the reference to his paper; and to Jim Stasheff for many detailed comments and suggestions. Above all I am grateful to the referee for pointing out an error in the original version and for making numerous suggestions which have greatly improved the paper.

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