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BIMONADICITY AND THE EXPLICIT BASIS PROPERTY

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ABSTRACT. Let $L \dashv R : \mathcal{X} \to \mathcal{Y}$ be an adjunction with R monadic and L comonadic. Denote the induced monad on $\mathcal{Y}$ by $\mathbf{M}$ and the induced comonad on $\mathcal{X}$ by $\mathbf{C}$. We characterize those $\mathbf{C}$ such that $\mathbf{M}$ satisfies the Explicit Basis property. We also discuss some new examples and results motivated by this characterization.

1. The *Explicit Basis* and *Redundant Coassociativity* properties

In May 2010 Lawvere conjectured that the unit law implies the associative law for comonads arising from EB monads as defined in [14]. The present paper grew out of the intention to understand that conjecture.

Let $\mathbf{C} = (C, \varepsilon, \delta)$ be a comonad on a category $\mathcal{X}$.

1.1. DEFINITION. A *pre-coalgebra* is a pair (X, s) where $s : X \to CX$ is a map in $\mathcal{X}$ such that the diagram below

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "id"'] & CX \arrow[d, "\varepsilon"] \\ & X \end{tikzcd}$$

commutes.

(Of course, pre-coalgebras are just 'coalgebras for the co-pointed endofunctor (C, ε)'; but we will need to consider both coalgebras and pre-coalgebras for the comonad $\mathbf{C}$ and, for this, it is more efficient to have a different name.)

Now fix an adjunction $L \dashv R : \mathcal{X} \to \mathcal{Y}$ with unit $\eta : Id \to RL$ and counit $\varepsilon : LR \to Id$. Let $C = LR : \mathcal{X} \to \mathcal{X}$ and denote the induced comonad on $\mathcal{X}$ by $\mathbf{C} = (C, \varepsilon, \delta)$.

Every pre-coalgebra (X, s) induces a coreflexive pair

$$\begin{tikzcd} RX \arrow[r, shift left=2, "\eta_R"] \arrow[r, shift right=2, "Rs"'] & RLRX \arrow[l, "R\varepsilon" description] \end{tikzcd} = RCX$$

in $\mathcal{Y}$, with $R\varepsilon$ as a common retraction of the subobjects η_R and Rs. (As in other contexts [10], it is possible to give in this case a useful meaning to the idea that the two subobjects united by $R\varepsilon$ are 'opposite'. See Section 3.)

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1.2. Definition. The *canonical restriction* of a pre-coalgebra (X, s) is the map in $\mathcal{Y}$ denoted by $\overline{s} : X_s \to RX$ and such that the square on the left below

$$\begin{tikzcd} X_s \arrow[r, "\overline{s}"] \arrow[d, "\overline{s}"'] & RX \arrow[d, "\eta_R"] \\ RX \arrow[r, "Rs"'] & RLRX \end{tikzcd} \qquad\qquad \begin{tikzcd} X_s \arrow[r, "\overline{s}"] & RX \arrow[r, shift left, "\eta"] \arrow[r, shift right, "Rs"'] & RLRX \end{tikzcd}$$

is a pullback (in $\mathcal{Y}$) and, equivalently, the fork on the right is an equalizer.

Canonical restrictions clearly exist if $\mathcal{Y}$ has equalizers or finite intersections of subobjects. Also, essentially due to Beck is the fact that existence of canonical restrictions is equivalent to the existence of a right adjoint to a canonical 'comparison' functor. (See Section 2 for details.)

Now let $\mathbf{M} = (M, \eta, \mu)$ be a monad on a category $\mathcal{Y}$, denote its category of algebras by $\mathcal{X}$, the forgetful functor by $R : \mathcal{X} \to \mathcal{Y}$ and its left adjoint by $L : \mathcal{Y} \to \mathcal{X}$, so that $M = RL$. Consider now the comonad $\mathbf{C} = (C, \varepsilon, \delta)$ induced by the adjunction $L \dashv R$ where $C = LR$. In this notation, the functor $C : \mathcal{X} \to \mathcal{X}$ assigns to each algebra $(A, a : MA \to A)$, the algebra $C(A, a) = (MA, \mu)$. The counit $\varepsilon : C(A, a) \to (A, a)$ is the canonical presentation $a : (MA, \mu) \to (A, a)$; and the comultiplication $\delta : C(A, a) \to C(C(A, a))$ is the canonical section $M\eta : (MA, \mu) \to (MMA, \mu_M)$ of the canonical presentation of (MA, μ).

1.3. Lemma. *Let $s : (A, a) \to C(A, a) = (MA, \mu)$ be a morphism in $\mathcal{X}$. Then the pair $((A, a), s)$ is a pre-coalgebra if and only if s is a section of the canonical presentation of (A, a) if and only if $as = id$. Moreover, such a section is a coalgebra if and only if the following diagram*

$$\begin{tikzcd} A \arrow[r, "s"] & MA \arrow[r, shift left, "M\eta"] \arrow[r, shift right, "Ms"'] & MMA \end{tikzcd}$$

commutes.

Proof. The axiom defining pre-coalgebras specializes to the commutativity of the following diagram

$$\begin{tikzcd} (A, a) \arrow[r, "s"] \arrow[dr, "id"'] & (MA, \mu) \arrow[d, "a"] \\ & (A, a) \end{tikzcd}$$

in $\mathcal{X}$. Such a pre-coalgebra $((A, a), s)$ is a coalgebra if and only if the diagram on the left below

$$\begin{tikzcd} (A, a) \arrow[r, "s"] \arrow[d, "s"'] & C(A, a) \arrow[d, "\delta"] \\ C(A, a) \arrow[r, "Cs"'] & C(C(A, a)) \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, "s"] \arrow[d, "s"'] & MA \arrow[d, "M\eta"] \\ MA \arrow[r, "Ms"'] & MMA \end{tikzcd}$$

commutes, if and only if the diagram on the right above commutes. ■

In other words, a pre-coalgebra $((A, a), s)$ is a coalgebra if and only if $s : A \to MA$ is a *strong Kleisli-idempotent* in the sense of Definition 4.2 in [14].

For a pre-coalgebra $((A, a), s)$ as above, its canonical restriction in the sense of Definition 1.2 coincides with the canonical restriction in the sense of Definition 2.1 in [14]. That is: the map $\overline{s} : A_s \to A$ making the following diagrams

$$\begin{tikzcd} A_s \arrow[r, "\overline{s}"] \arrow[d, "\overline{s}"'] & A \arrow[d, "\eta"] \\ A \arrow[r, "s"'] & MA \end{tikzcd} \qquad\qquad \begin{tikzcd} A_s \arrow[r, "\overline{s}"] & A \arrow[r, shift left, "\eta"] \arrow[r, shift right, "s"'] & MA \end{tikzcd}$$

a pullback and an equalizer respectively.

1.4. DEFINITION. The monad $\mathbf{M} = (M, \eta, \mu)$ is said to satisfy the *Explicit Basis* property if for every pre-coalgebra $((A, a), s)$ as above the canonical restriction exists and the map

$$MA_s \xrightarrow{M\overline{s}} MA \xrightarrow{a} A$$

is an iso.

This is one form of the Explicit Basis (EB) property (see Definition 2.4 in [14]). It seems worth recalling some relevant terminology. If $f : U \to A$ is a morphism in $\mathcal{Y}$, the composition $a(Mf) : MU \to A$ underlies a morphism $(MU, \mu) \to (A, a)$ of algebras. The map f is called *independent* (w.r.t. the algebra (A, a)) if the induced map $a(Mf) : MU \to A$ is mono; it is called *spanning* if $a(Mf) : MU \to A$ is regular epi; and it is called a *basis* if it is both independent and spanning. In this terminology $\mathbf{M}$ satisfies the EB property if for every algebra (A, a) and section $s : (A, a) \to (MA, \mu)$ of the canonical presentation, the canonical restriction of s is a basis for (A, a).

Proposition 2.5 in [14] shows that the EB property is a strengthening of the classical condition ‘projectives are free’. Lawvere suggested the following informal explanation: if one understands s as a ‘proof that (A, a) is projective’ then the EB property gives a simple algorithm that produces a system of free generators from any given proof of projectivity.

What does not appear in [14] is the role of the comonadic notions that we have used in the paragraphs above. These became evident only after Lawvere formulated the conjecture mentioned in the first paragraph. We can now start to explain it. As in the beginning, let $\mathbf{C} = (C, \varepsilon, \delta)$ be a comonad on a category $\mathcal{X}$.

1.5. DEFINITION. A *map* $f : (X, s) \to (X', s')$ of pre-coalgebras is a map $f : X \to X'$ in $\mathcal{X}$ such that the diagram below

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & CX \arrow[d, "Cf"] \\ X' \arrow[r, "s'"'] & CX' \end{tikzcd}$$

commutes.

Let $\underline{\mathcal{X}_{\mathbf{C}}}$ be the category of pre-coalgebras and morphisms between them. Clearly, the category $\mathcal{X}_{\mathbf{C}}$ of $\mathbf{C}$-coalgebras is a full subcategory $\mathcal{X}_{\mathbf{C}} \to \underline{\mathcal{X}_{\mathbf{C}}}$.

1.6. DEFINITION. We say that $\mathbf{C}$ satisfies the *Redundant Coassociativity* property if the embedding $\mathcal{X}_{\mathbf{C}} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence.

Immediate from Lemma 1.3 is the following.

1.7. COROLLARY. *Let $\mathbf{M} = (M, \eta, \mu)$ be a monad on $\mathcal{Y}$ and denote the induced comonad on the category $\mathcal{Y}^{\mathbf{M}}$ of algebras by $\mathbf{C}$. Then $\mathbf{C}$ satisfies Redundant Coassociativity if and only if for every algebra (A, a) and every section $s : (A, a) \to (MA, \mu)$ of the canonical presentation of (A, a), the map $s : A \to MA$ is a strong Kleisli idempotent.*

Corollary 1.7, together with Proposition 4.5 in [14], implies that if M preserves intersections then Redundant Coassociativity is equivalent to the Explicit Basis property. This is already a strong connection, but we will show that there is a more elegant relation between the two properties which, in particular, explains the existence of canonical restrictions via a right adjoint.

The examples discussed in [14] suggest that it is not uncommon for EB monads to have an intersection-preserving and conservative (i.e. faithful and iso-reflecting) underlying functor. Since the forgetful functor from algebras is conservative and creates limits, the free-algebra functor is, in the examples, intersection-preserving and conservative. Such left adjoints are comonadic by Beck's Theorem. So the Eilenberg-Moore adjunctions induced by EB monads are, in many cases, bimonadic in the following sense.

1.8. DEFINITION. An adjunction $L \dashv R : \mathcal{X} \to \mathcal{Y}$ is called *bimonadic* if $R : \mathcal{X} \to \mathcal{Y}$ is monadic and $L : \mathcal{Y} \to \mathcal{X}$ is comonadic.

The main result in the paper (to be proved in Section 2) is an extension of Beck's comonadicity theorem and implies the following.

1.9. COROLLARY. *Let $L \dashv R : \mathcal{X} \to \mathcal{Y}$ be a bimonadic adjunction. If we denote the induced monad on $\mathcal{Y}$ by $\mathbf{M}$ and the induced comonad on $\mathcal{X}$ by $\mathbf{C}$ then $\mathbf{M}$ is EB if and only if $\mathbf{C}$ satisfies the Redundant Coassociativity property.*

We stress that bimonadicity of $L \dashv R : \mathcal{X} \to \mathcal{Y}$ does not imply that the induced monad on $\mathcal{Y}$ is EB. The examples of non-EB monads over $\mathbf{Set}$ discussed in [14] induce bimonadic adjunctions by Barr's characterization in [1]. As another example, consider the bimonadic adjunction $F \dashv U : \mathbf{Cat} \to \mathbf{Set}^{\rightrightarrows}$ where U assigns, to each small category, its underlying non-reflexive graph. The induced monad on $\mathbf{Set}^{\rightrightarrows}$ is not EB, as shown in Example 5.5 in [14]. This example should be contrasted with the conservative EB monad determined by the forgetful $U : \mathbf{Cat} \to \widehat{\Delta_1}$ from categories to reflexive graphs (Corollary 5.4 loc. cit.).

In Sections 4 and 5 we discuss two sources of EB monads that were recognized as such only after the comonadic perspective explained in this paper was clear. The last sections present two simple applications showing how the EB property may be used in the contexts where it holds: in Section 7 we give an alternative proof of the characterization of modular categories by Carboni and Janelidze and, in Section 8, we prove one of the basic facts about Peano algebras.

We assume familiarity with [14] and with Beck's (Co)Monadicity Theorem [2]. See also Theorem 2.5 in [16] for a sharp statement distinguishing the different properties of the comparison adjunction. Also in [16], a monad $\mathbf{M}$ on a category $\mathcal{Y}$ is said to be of *effective descent type* if the free-algebra functor $\mathcal{Y} \to \mathcal{Y}^{\mathbf{M}}$ is comonadic.

2. The extended comparison

Fix an adjunction $L \dashv R : \mathcal{X} \to \mathcal{Y}$ with unit $\eta : Id \to RL$ and counit $\varepsilon : LR \to Id$. Let $C = LR : \mathcal{X} \to \mathcal{X}$ and denote the induced comonad on $\mathcal{X}$ by $\mathbf{C} = (C, \varepsilon, \delta)$. The standard comparison functor $K : \mathcal{Y} \to \mathcal{X}_{\mathbf{C}}$ assigns, to each object Y in $\mathcal{Y}$, the coalgebra $(LY, L\eta : LY \to CLY)$ and makes the following diagram

$$\begin{tikzcd} & \mathcal{X} & \\ \mathcal{Y} \arrow[ur, "L"] \arrow[r, "K"] \arrow[rr, bend right, "\underline{K}"'] & \mathcal{X}_{\mathbf{C}} \arrow[u] \arrow[r] & \underline{\mathcal{X}_{\mathbf{C}}} \arrow[ul] \end{tikzcd}$$

commute, where $\mathcal{X}_{\mathbf{C}} \to \mathcal{X}$ and $\underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{X}$ are the obvious forgetful functors, and the functor $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is the composition of K with the embedding $\mathcal{X}_{\mathbf{C}} \to \underline{\mathcal{X}_{\mathbf{C}}}$. The next result is then obvious.

2.1. Lemma. *If L is comonadic and Redundant Coassociativity holds then the extended comparison $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence.*

We will prove that the converse holds and relate the fact with the EB property. First, following Beck, we ask when does $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ have a right adjoint. The answer is in Proposition 2.3, which needs the next extension of Lemma 2.10 in [14].

2.2. Lemma. *Let $f : Y \to RX$ and $g : LY \to X$ be adjuncts. For any pre-coalgebra (X, s) the following are equivalent:*

1. *the map g underlies a pre-coalgebra map $\underline{K}Y = (LY, L\eta) \to (X, s)$;*

2. *the diagram*

$$\begin{tikzcd} LRX \arrow[r, "\varepsilon"] \arrow[r, phantom, "="', shift right=3ex] & X \arrow[d, "s"] \\ LY \arrow[u, "Lf"] \arrow[ur, "g"] \arrow[r, "Lf"'] & LRX \end{tikzcd}$$

commutes (i.e. $Lf = s\varepsilon(Lf)$ or, equivalently, $sg = Lf$);

3. *the diagram*

$$\begin{tikzcd} Y \arrow[r, "f"] & RX \arrow[r, shift left, "\eta_R"] \arrow[r, shift right, "Rs"'] & RLRX \end{tikzcd}$$

commutes.

Proof. If the third item holds the diagram below

$$\begin{tikzcd} LY \arrow[r, "Lf"] & LRX \arrow[r, shift left, "L\eta_R"] \arrow[r, shift right, "LRs"'] \arrow[d, "\varepsilon"'] & LRLRX \arrow[d, "\varepsilon_{LR}"] \\ & X \arrow[r, "s"'] & LRX \end{tikzcd}$$

shows that $s\varepsilon(Lf) = \varepsilon(L\eta)(Lf) = Lf$, so the second item holds. If this is the case, the next diagram

$$\begin{tikzcd} LY \arrow[r, "Lf"] \arrow[d, "L\eta"'] \arrow[drr, "Lf" description] & LRX \arrow[r, "\varepsilon"] & X \arrow[d, "s"] \\ LRLY \arrow[r, "LRLf"'] & LRLRX \arrow[r, "LR\varepsilon"'] & LRX \end{tikzcd}$$

proves that $\varepsilon(Lf) = g : (LY, L\eta) \to (X, s)$, so the first item holds. Finally, if we assume that the rectangle above commutes then, applying R and precomposing with the unit $\eta : Y \to RLY$ it is easy to confirm that the third item holds. ∎

Apart from its mathematical content, the next result shows that the condition 'canonical restrictions exist' used in [14] is not ad-hoc.

2.3. Proposition. *The functor $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ has a right adjoint if and only if canonical restrictions exist. Moreover, in this case, the right adjoint maps each pre-coalgebra (X, s) to the domain of its canonical restriction $\overline{s} : X_s \to RX$.*

Proof. Assume first that the functor $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ has a right adjoint that we denote by $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$. Also, let us denote the counit of $\underline{K} \dashv \underline{N}$ by $\xi : \underline{K}(\underline{N}(X, s)) \to (X, s)$. For brevity we temporarily denote $\underline{N}(X, s)$ by S, an object in $\mathcal{Y}$. In this notation $\xi : (LS, L\eta) \to (X, s)$. The underlying map $\xi : LS \to X$ has a transposition that we denote by $\tau : S \to RX$. We claim that the following diagram

$$\begin{tikzcd} S \arrow[r, "\tau"] & RX \arrow[r, shift left, "\eta_R"] \arrow[r, shift right, "Rs"'] & RLRX \end{tikzcd}$$

is an equalizer in $\mathcal{Y}$. For this, assume that $f : Y \to RX$ is such that $\eta f = (Rs)f$. By Lemma 2.2, this is equivalent to the transposition $g : LY \to X$ of f underlying a map $(LY, L\eta) \to (X, s)$ of pre-coalgebras. By hypothesis, there exists a unique map $g' : Y \to S$ such that $\xi(\underline{K}g') = g$. That is, $\xi(Lg') = g : LY \to X$ in $\mathcal{Y}$. Transposition shows that g' is the unique map such that $\tau g' = f : Y \to RX$. So the fork above is indeed an equalizer and this shows that the canonical restriction of (X, s) exists.

Conversely, assuming that canonical restrictions exist, we now show that the map $\varepsilon(L\overline{s}) : \underline{K}X_s \to (X, s)$ is universal from $\underline{K}$ to (X, s). For this let $g : \underline{K}Y \to (X, s)$ be a map in $\underline{\mathcal{X}_{\mathbf{C}}}$. By Lemma 2.2 the transposition $f : Y \to RX$ of $g : LY \to X$ factors through the canonical restriction $\overline{s} : X_s \to X$, say, via a map $f' : Y \to X_s$. The next diagram

$$\begin{tikzcd} LY \arrow[d, "Lf'"'] \arrow[dr, "Lf"] \arrow[drr, "g"] & & \\ LX_s \arrow[r, "L\overline{s}"'] & LRX \arrow[r, "\varepsilon"'] & X \end{tikzcd}$$

completes the proof. ∎

The right adjoint to $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ will be denoted by $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$. The composition

$$\mathcal{X}_{\mathbf{C}} \longrightarrow \underline{\mathcal{X}_{\mathbf{C}}} \xrightarrow{\underline{N}} \mathcal{Y}$$

will be denoted by N and is obviously right adjoint to $K : \mathcal{Y} \to \mathcal{X}_{\mathbf{C}}$. It is therefore the well-known adjoint in Beck's Theorem. Part of Beck's result characterizes coreflexivity of the adjunction $K \dashv N$ in terms of regularity of the unit η. This easily extends as follows.

2.4. Corollary. *The functor $K : \mathcal{Y} \to \mathcal{X}_{\mathbf{C}}$ is full and faithful if and only if $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is. If, moreover, $\underline{K}$ has a right adjoint then the above are also equivalent to the following diagram being an equalizer*

$$Y \xrightarrow{\eta} MY \underset{M\eta}{\overset{\eta_M}{\rightrightarrows}} MMY$$

for every Y in $\mathcal{Y}$, where $M = RL : \mathcal{Y} \to \mathcal{Y}$.

Proof. The first equivalence holds because $\mathcal{X}_{\mathbf{C}} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is full and faithful. If $\underline{K}$ has a right adjoint then $\underline{K}$ full and faithful means that the unit $\lambda : Id \to \underline{NK}$ is an iso. Since $\underline{K}Y = (LY, L\eta)$, this is equivalent to the vertical map below

$$\begin{tikzcd} Y \arrow[d, "\lambda"'] \arrow[dr, "\eta"] & & \\ \underline{N}(\underline{K}Y) \arrow[r, "\overline{L\eta}"'] & RLY \arrow[r, shift left, "\eta"] \arrow[r, shift right, "RL\eta"'] & RLRLY \end{tikzcd}$$

being an iso. (The relation between K and the equalizer diagram is, of course, part of Beck's comonadicity theorem.) ∎

We now discuss the corresponding 'reflection' conditions.

2.5. Lemma. *If $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ exists then $\underline{N}$ is full and faithful if and only if for every pre-coalgebra (X, s) the morphism*

$$LX_s \xrightarrow{L\overline{s}} LRX \xrightarrow{\varepsilon} X$$

is an iso in $\mathcal{X}$. Also, $N : \mathcal{X}_{\mathbf{C}} \to \mathcal{Y}$ is full and faithful if and only if for every coalgebra (X, s) the following diagram

$$LX_s \xrightarrow[L\overline{s}]{} LRX \underset{LRs}{\overset{L\eta_R}{\rightrightarrows}} LRLRX$$

is an equalizer.

Proof. By Proposition 2.3, the transposition of $id : \underline{N}(X,s) \to \underline{N}(X,s)$ (i.e. the counit $\underline{KN}(X,s) \to (X,s)$) is determined by the map $\varepsilon(L\overline{s}) : LX_s \to X$.

Also, we have already mentioned that if $\underline{N}$ exists then its restriction to $\mathcal{X}_{\mathbf{C}}$ is N. So the final part is a fraction of Beck's theorem. ∎

In other words $N : \mathcal{X}_{\mathbf{C}} \to \mathcal{Y}$ is full and faithful if and only if $L : \mathcal{Y} \to \mathcal{X}$ 'respects' canonical restrictions. In contrast, we do not know any conditions on L alone that imply reflexivity of the adjunction $\underline{K} \dashv \underline{N}$.

2.6. Proposition. *If the functor $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ exists then it is full and faithful if and only if both $N : \mathcal{X}_{\mathbf{C}} \to \mathcal{Y}$ is full and faithful and Redundant Coassociativity holds.*

Proof. One direction is trivial because if N is fully faithful and the vertical functor below is an equivalence

$$\begin{tikzcd} \mathcal{Y} & \mathcal{X}_{\mathbf{C}} \arrow[l, "N"'] \arrow[d, "\text{full and faithful}"] \\ & \underline{\mathcal{X}_{\mathbf{C}}} \arrow[ul, "\underline{N}"] \end{tikzcd}$$

then $\underline{N}$ is fully faithful. Conversely, assume that $\underline{N}$ is full and faithful. Then, clearly, so is N. So all we need to prove is that Redundant Coassociativity holds. Let (X,s) be a pre-coalgebra. Lemma 2.2 and the definition of $\overline{s}$ imply that the 'long fork' in the diagram below

$$\begin{tikzcd} LX_s \arrow[r, "L\overline{s}"'] \arrow[rrr, bend left, "L\overline{s}"] & LRX \arrow[r, "\varepsilon"'] & X \arrow[r, "s"'] & LRX \arrow[r, shift left, "L\eta_R"] \arrow[r, shift right, "LRs"'] & LRLRX \end{tikzcd}$$

commutes. As $\varepsilon(L\overline{s})$ is epi by Lemma 2.5, $(L\eta_R)s = (LRs)s$, which means that the pre-coalgebra (X,s) is coassociative. ∎

Summarizing the above discussion we obtain the following result.

2.7. Corollary. *The extended comparison $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence if and only if $L : \mathcal{Y} \to \mathcal{X}$ is comonadic and Redundant Coassociativity holds.*

Proof. One direction is just Lemma 2.1. For the converse assume that $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence. Then it is full and faithful and it has a full and faithful right adjoint $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$. Corollary 2.4 implies that $K : \mathcal{Y} \to \mathcal{X}_{\mathbf{C}}$ is full and faithful. Proposition 2.6 implies that $N : \mathcal{X}_{\mathbf{C}} \to \mathcal{Y}$ is full and faithful and Redundant Coassociativity holds. Then $K \dashv N$ is an equivalence (so L is comonadic) and Redundant Coassociativity holds. ∎

Now let $M = RL : \mathcal{Y} \to \mathcal{Y}$ and $\mathbf{M} = (M, \eta, \mu)$ be the monad on $\mathcal{Y}$ induced by the adjunction $L \dashv R$. If $R : \mathcal{X} \to \mathcal{Y}$ is monadic then we can work with $\mathcal{X}$ is if it was the category $\mathcal{Y}^{\mathbf{M}}$ of $\mathbf{M}$-algebras. From this perspective, the functor $C : \mathcal{X} \to \mathcal{X}$ assigns, to each algebra $(A, a : MA \to A)$, the algebra $C(A,a) = (MA, \mu)$.

2.8. Corollary. *If $R : \mathcal{X} \to \mathcal{Y}$ is monadic the following hold:*

1. *$\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ exists and is full and faithful if and only if the EB property holds.*
2. *$\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence if and only if EB holds and M reflects isos.*

Proof. Lemma 2.5 applied in the present context says that $\underline{N}$ is full and faithful if and only if for every algebra (A, a) and section $s : (A, a) \to (MA, \mu)$ of the canonical presentation of (A, a), the map

$$MA_s \xrightarrow{\overline{s}} MA \xrightarrow{a} A$$

is an iso; but this is the EB property (Definition 1.4). Finally, for general reasons, $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence if and only if it reflects isos and it has a full and faithful right adjoint $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$; but the concrete definition of $\underline{K}$ implies that it reflects isos if and only if $L : \mathcal{Y} \to \mathcal{X}$ does. ∎

Corollaries 2.7 and 2.8 imply the the next variation of Beck's comonadicity theorem.

2.9. Theorem. *If $R : \mathcal{X} \to \mathcal{Y}$ is monadic the following are equivalent:*

1. *$\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ is an equivalence.*
2. *$L : \mathcal{Y} \to \mathcal{X}$ is comonadic and Redundant Coassociativity holds.*
3. *The induced monad on $\mathcal{Y}$ reflects isos and satisfies the Explicit Basis property.*

Of course, in this case, the adjunction $L \dashv R : \mathcal{X} \to \mathcal{Y}$ is bimonadic.

Corollary 1.9 in the introduction follows.

Also, as a corollary of Proposition 2.6, together with Lemma 2.5, we obtain the next more directly applicable 'crude' version.

2.10. Corollary. *If $\mathcal{Y}$ has finite intersections and L preserves them then $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ is full and faithful if and only if Redundant Coassociativity holds.*

Again, as a by product of the results here we can reprove Proposition 4.5 in [14] which was mentioned after Corollary 1.7.

2.11. Corollary. *Let $\mathbf{M} = (M, \eta, \mu)$ be a monad on $\mathcal{Y}$. If $\mathcal{Y}$ has finite intersections of subobjects and M preserves them then the following are equivalent:*

1. *$\mathbf{M}$ satisfies the Explicit Basis property.*
2. *For every algebra (A, a) and section $s : (A, a) \to (MA, \mu)$ of the canonical presentation of (A, a), $s : A \to MA$ is a strong Kleisli idempotent.*

Proof. Let $\mathcal{X}$ be the category of $\mathbf{M}$-algebras and $\mathbf{C}$ be the comonad on $\mathcal{X}$ induced by the monadic adjunction $\mathcal{X} \to \mathcal{Y}$. By Corollary 2.8, $\mathbf{M}$ is EB if and only if $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ is full and faithful. In turn, this is equivalent to Redundant Coassociativity of $\mathbf{C}$ by Corollary 2.10. Corollary 1.7 completes the proof. ∎

To end this section we introduce a piece of convenient terminology.

2.12. Definition. A monad satisfying the conditions of the third item in Theorem 2.9 will be called a *CEB* monad.

Every CEB monad has a conservative underlying functor.

3. A definite notion of 'opposite'

In this section we assume that $\mathcal{Y}$ has equalizers. Fix an adjunction $L \dashv R : \mathcal{X} \to \mathcal{Y}$ with unit $\eta : Id \to RL$ and counit $\varepsilon : LR \to Id$. As mentioned in the beginning every pre-coalgebra (X, s) for the comonad induced by $L \dashv R$ determines a coreflexive pair

$$\begin{tikzcd} RX \arrow[r, shift left=2, "\eta_R"] \arrow[r, shift right=2, "Rs"'] & RLRX \arrow[l, "R\varepsilon" description] \end{tikzcd}$$

in $\mathcal{Y}$, with $R\varepsilon$ as a common retraction of the subobjects η_R and Rs. In the terminology of [10], $R\varepsilon : RLRX \to RX$ is a *Unity and Identity (UI)* of the 'opposites' η_R and Rs. The terminology is justified, among other reasons, because it is usually possible to give "a useful meaning to the idea that the two subobjects united by a UI are 'opposite' " (see p. 169 loc. cit.). In this section we describe a definite notion of 'opposite' relevant to UIs above. Notice that an opposition is already manifest in the fact that while one of the sections is in the image of R the other is not, unless the diagram is trivial. Indeed, if η_R was equal to $R\gamma$ for some $\gamma : X \to LRX$ then $\varepsilon : LRX \to X$ would be a section of γ and so an iso, with inverse $s = \gamma$.

Let us stress that we are not understanding 'opposite' as the two sections satisfying respective conditions whose conjunction is inconsistent. Instead, the idea we have just described is that the conjunction implies a certain extreme situation. This practice of using simple negative phrases to signify 'implying a trivial case' of course requires that it be explained in each situation what the trivial case is. For example, one might say that a Boolean atom is an object with no subobjects. It is also worth mentioning that Läuchli's completeness result for intuitionistic first order logic relies on the fact that he understood negation in the spirit suggested above: the usual definition of $\neg p$ is $p \Rightarrow \perp$ but in various contexts it may be more appropriate to replace $\perp$ with some other value. (See [15] and references therein.)

Let $f : RX \to RY$ be a map in $\mathcal{Y}$. The diagram

$$\begin{tikzcd} RLRX \arrow[r, "RLf"] \arrow[d, "R\varepsilon"'] & RLRY \arrow[d, "R\varepsilon"] \\ RX \arrow[r, "f"'] & RY \end{tikzcd}$$

need not commute. In any case, we can take the equalizer of the two compositions $RLRX \to RY$.

3.1. DEFINITION. The $(L \dashv R)$-*core* of f is the equalizer of the maps below

$$RLRX \xrightarrow{R\varepsilon} RX \xrightarrow{f} RY \qquad\qquad RLRX \xrightarrow{RLf} RLRY \xrightarrow{R\varepsilon} RY$$

from $RLRX$ to RY in $\mathcal{Y}$.

Since the adjunction is fixed we will refer to the *core* of f instead of its $(L \dashv R)$-core. Now, how small can it be?

3.2. LEMMA. *The unit $\eta_R : RX \to RLRX$ factors through the core of f.*

PROOF. The diagram below

$$\begin{tikzcd}
RX \arrow[r, "\eta_R"] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & RLRX \arrow[r, "R\varepsilon"] \arrow[d, "RLf"'] & RX \arrow[d, "f"] \\
RY \arrow[r, "\eta_R"] \arrow[rr, bend right, "id"'] & RLRX \arrow[r, "R\varepsilon"] & RY
\end{tikzcd}$$

shows that $(R\varepsilon)(RLf)\eta_R = f(R\varepsilon)\eta_R : RX \to RY$. (Bare in mind that the square on the right above does not necessarily commute.) ■

We are therefore led to the following.

3.3. DEFINITION. The map $f : RX \to RY$ is *reluctant* (w.r.t. X and Y) if its core is $\eta_R : RX \to RLRX$.

The following simple fact suggests that it is fair to say that reluctant maps are not in the image of R.

3.4. LEMMA. *Let $g : X \to Y$ be a map in $\mathcal{X}$. Then $Rg : RX \to RY$ is reluctant if and only if $\eta_R : RX \to RLRX$ is an iso.*

PROOF. The maps $g\varepsilon_X, \varepsilon_Y(LRg) : RLX \to Y$ are equal by naturality, so the core of Rg is an iso. If Rg is reluctant it means that $\eta_R : RX \to RLRX$ is an iso. On the other hand, if $\eta_R : RX \to RLRX$ is an iso then it is the core of Rg by Lemma 3.2. ■

In other words, if f is in the image of R and is reluctant then its domain is of a very restricted form. Notice that Lemma 3.2 implies that if X is such that $\eta_R : RX \to RLRX$ is an iso then every map $f : RX \to RY$ is reluctant.

3.5. LEMMA. *The unit $\eta_R : RX \to RLRX$ is reluctant w.r.t. X and LRX.*

PROOF. The core of $\eta_R : RX \to RLRX$ is the equalizer the maps

$$RLRX \xrightarrow{R\varepsilon} RX \xrightarrow{\eta_R} RLRX \qquad\qquad \begin{tikzcd}
RLRX \arrow[r, "RL\eta_R"] \arrow[dr, bend right, "id"'] & RLRLRX \arrow[d, "R\varepsilon_{LR}"] \\
 & RLRX
\end{tikzcd}$$

so it must be the unit $\eta_R : RX \to RLRX$ because $(R\varepsilon_X)\eta_{RX} = id_{RX}$. ■

So, in the coreflexive pair determined by a pre-coalgebra $s : X \to LRX$, the sections involved are opposite in the sense that one is in the image of R and the other is reluctant.

Assume from now on that R is monadic and let $\mathbf{M} = (M, \eta, \mu)$ be the induced monad on $\mathcal{Y}$. In this case, the material above admits a more concrete intuition. Let $f : A \to B$ be a map in $\mathcal{Y}$ and let (A, a) and (B, b) be $\mathbf{M}$-algebras. The diagram

$$\begin{tikzcd} MA \arrow[r, "Mf"] \arrow[d, "a"'] & MB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

need not commute. The core of f (w.r.t. (A, a) and (B, b)) is the equalizer of the maps below

$$MA \xrightarrow{a} A \xrightarrow{f} B \qquad\qquad MA \xrightarrow{Mf} MB \xrightarrow{b} B$$

from MA to B in $\mathcal{Y}$. So f underlies an algebra map $(A, a) \to (B, b)$ if and only if its core is an iso. In this context, a reluctant map may be called *reluctantly homomorphic* or *strongly non-homomorphic*.

In many cases, strongly non-homomorphic homomorphisms don't exist. For example, for the monad on **Set** induced by any consistent algebraic theory with a constant, there are no objects A such that $\eta : A \to MA$ is an iso. On the other hand, for the monad determined by the algebraic category of monoid actions of some non-trivial monoid, $A = \emptyset$ is the only set such that $\eta : A \to MA$ is an iso.

4. Descent data induce CEB monads

In this section we show that the typical examples of monads arising in 'Monadic Descent Theory' [8] are CEB (Definition 2.12). Fix a category $\mathcal{E}$ with finite limits. For any object I of $\mathcal{E}$ the forgetful $\mathcal{E}/I \to \mathcal{E}$ is comonadic and it is not difficult to check that the resulting comonad on $\mathcal{E}$ satisfies the Redundant Coassociativity property. Applying this observation to a slice category one obtains the following result which we learned from R. Wood (who said that probably others had observed it before).

4.1. Proposition. *For any map $p : I \to J$ in $\mathcal{E}$ the comonad on $\mathcal{E}/J$ determined by $\Sigma_p \dashv p^* : \mathcal{E}/J \to \mathcal{E}/I$ satisfies the Redundant Coassociativity property.*

Denote the induced monad on $\mathcal{E}/I$ by $\mathbf{M} = (M, \eta, \mu)$. It follows that if p is *effective descent* (i.e. p^* is monadic) then $\mathbf{M}$ is EB by Corollary 1.9. We now show that $\mathbf{M}$ is EB regardless of whether p is effective descent or not.

The functor M maps $(X, x : X \to I)$ to $(I \times X, \pi_0)$. The unit $\eta : (X, x) \to M(X, x)$ is defined as the horizontal map on the left below

$$\begin{tikzcd} X \arrow[r, "\langle x, id\rangle"] \arrow[dr, "x"'] & I \times X \arrow[d, "\pi_0"] \\ & I \end{tikzcd} \qquad\qquad \begin{tikzcd} I \times I \times X \arrow[r, "\pi_0 \times X"] \arrow[dr, "\pi_0"'] & I \times X \arrow[d, "\pi_0"] \\ & I \end{tikzcd}$$

and the multiplication $\mu : MM(X,x) \to M(X,x)$ is defined as on the right above. Algebras for this monad are sometimes called *descent data*.

4.2. LEMMA. *An algebra for* $\mathbf{M}$ *is a triple* $((X,x),a)$ *with* $a : M(X,x) \to (X,x)$ *so that the following diagram*

$$\begin{tikzcd} I \times X \arrow[r, "a"] \arrow[dr, "\pi_0"'] & X \arrow[d, "x"] \\ & I \end{tikzcd}$$

commutes, and such that unit and associativity hold, so that the following

$$\begin{tikzcd} X \arrow[r, "\langle x, id \rangle"] \arrow[dr, "id"'] & I \times X \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} I \times I \times X \arrow[r, "I \times a"] \arrow[d, "\pi_0 \times X"'] & I \times X \arrow[d, "a"] \\ I \times X \arrow[r, "a"'] & X \end{tikzcd}$$

commute.

A section $s : ((X,x),a) \to (M(X,x),\mu)$ for the canonical presentation of such an algebra is determined by a map $s : X \to I \times X$ in $\mathcal{E}$ such that the following diagrams

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "x"'] & I \times X \arrow[d, "\pi_0"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "id"'] & I \times X \arrow[r, "I \times s"] \arrow[d, "a"] & I \times I \times X \arrow[d, "\pi_0 \times X"] \\ & X \arrow[r, "s"'] & I \times X \end{tikzcd}$$

commute.

4.3. LEMMA. *For* $s : X \to I \times X$ *as above, the composition*

$$X \xrightarrow{s} I \times X \xrightarrow{\pi_1} X$$

is idempotent.

PROOF. Denote the composition $\pi_1 s$ by $e : X \to X$. First we show that the following diagram

$$\begin{tikzcd} I \times X \arrow[r, "\pi_1"] \arrow[d, "a"'] & X \arrow[d, "e"] \\ X \arrow[r, "e"'] & X \end{tikzcd}$$

commutes. Indeed, the square on the left below commutes

$$\begin{tikzcd} & & X \arrow[d, "s"] \arrow[dd, bend left=60, "e"] \\ I \times X \arrow[urr, bend left, "\pi_1"] \arrow[r, "I \times s"] \arrow[d, "a"'] & I \times I \times X \arrow[r, "\pi_1 \times X"] \arrow[d, "\pi_0 \times X"] & I \times X \arrow[d, "\pi_1"] \\ X \arrow[r, "s"] \arrow[rr, bend right, "e"'] & I \times X \arrow[r, "\pi_1"] & X \end{tikzcd}$$

because s is an algebra map and the square on the right commutes trivially. Finally, to prove that e is idempotent, pre-compose with s and calculate as below

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[drr, bend left, phantom] \arrow[dr, "id"'] \arrow[rr, bend left, "e"] & I \times X \arrow[r, "\pi_1"] \arrow[d, "a"] & X \arrow[d, "e"] \\ & X \arrow[r, "e"'] & X \end{tikzcd}$$

using that s is a section of a. ∎

4.4. Proposition. *The monad* $\mathbf{M}$ *on* $\mathcal{E}/I$ *is CEB.*

Proof. It is easy to check that M reflects isos. To prove that the EB property holds we use Corollary 2.11. Finite intersections exist in $\mathcal{E}/I$ because we have assumed that pullbacks exist in $\mathcal{E}$ and the forgetful $\mathcal{E}/I \to \mathcal{E}$ creates them. Also for this reason, and because products preserve pullbacks, M preserves pullbacks and, in particular, intersections. So we are left to prove that sections of canonical presentations are strong Kleisli idempotents. For this, let $((X, x), a)$ be an algebra and s a section for its canonical presentation as above. To prove that s is a strong Kleisli idempotent we must check that the fork below

$$\begin{tikzcd} X \arrow[r, "s"] & I \times X \arrow[r, shift left, "I\times\langle x,id\rangle"] \arrow[r, shift right, "I\times s"'] & I \times I \times X \end{tikzcd}$$

commutes in $\mathcal{E}$. If we post-compose with $\pi_0 : I \times I \times X \to I$ then, clearly, the resulting maps $X \to I$ are equal (to $\pi_0 s$). To prove that the two maps are equal after post-composing with $\pi_1 : I \times I \times X \to I$, just observe that the following

$$\begin{tikzcd} X \arrow[r, "s"] & I \times X \arrow[rr, "I\times s"] \arrow[d, "\pi_1"'] & & I \times I \times X \arrow[d, "\pi_1"] \\ & X \arrow[r, "s"] \arrow[rr, bend right, "x"'] & I \times X \arrow[r, "\pi_0"] & I \end{tikzcd}$$

commutes because $s : (X, x) \to (I \times X, \pi_0)$ in $\mathcal{E}/I$. Finally, use Lemma 4.3 to prove that the two maps are the same after composing with $\pi_2 : I \times I \times X \to X$. ∎

The usual trick with slices implies the promised result (which was suggested by a talk by R. Rosebrugh based on [9] in a Seminar organized by R. F. C. Walters at Como in 2010).

4.5. Corollary. *If* $\mathcal{E}$ *has pullbacks then, for any* $p : I \to J$ *in* $\mathcal{E}$*, the monad on* $\mathcal{E}/I$ *determined by* $\Sigma_p \dashv p^* : \mathcal{E}/J \to \mathcal{E}/I$ *is CEB.*

One of the main results in [16] states that a functor is comonadic if and only if the associated Eilenberg-Moore adjunction is bimonadic. This result can be applied to $\Sigma_p : \mathcal{E}/I \to \mathcal{E}/J$ but we don't know if the Redundant Coassociativity proved in Proposition 4.1 'lifts' to the induced Eilenberg-Moore adjunction in general.

5. Extensivity, additivity and modularity

Let $\mathcal{D}$ be a category with finite coproducts.

5.1. Definition. An object D in $\mathcal{D}$ is called *detachable* if the monad induced by the right adjoint $D/\mathcal{D} \to \mathcal{D}$ is CEB.

Let us be more explicit.

5.2. Lemma. *D is detachable if and only if the following hold:*

1. *For every $a : D \to A$ and $s : A \to D + A$ such that the following diagrams*

$$\begin{tikzcd} D \arrow[r, "a"] \arrow[dr, "in_0"'] & A \arrow[d, "s"] \\ & D + A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "s"] \arrow[dr, "id"'] & D + A \arrow[d, "{[a,id]}"] \\ & A \end{tikzcd}$$

commute, the equalizer/intersection

$$\begin{tikzcd} A_s \arrow[r, "\overline{s}"] & A \arrow[r, shift left, "in_1"] \arrow[r, shift right, "s"'] & D + A \end{tikzcd}$$

exists and the map $[a, \overline{s}] : D + A_s \to A$ is an iso.

2. *The following diagram*

$$\begin{tikzcd} X \arrow[r, "in_1"] \arrow[d, "in_1"'] & D + X \arrow[d, "in_1"] \\ D + X \arrow[r, "D+in_1"'] & D + (D + X) \end{tikzcd}$$

is a pullback for every X in $\mathcal{D}$.

Proof. Let $\mathcal{X} = D/\mathcal{D} \to \mathcal{D} = \mathcal{Y}$ be the forgetful functor. The induced monad is CEB if and only if both $\underline{K} : \mathcal{Y} \to \underline{\mathcal{X}_{\mathbf{C}}}$ and $\underline{N} : \underline{\mathcal{X}_{\mathbf{C}}} \to \mathcal{Y}$ are full and faithful. By Corollary 2.8, $\underline{N}$ is full and faithful if and only if the EB property holds. In turn the EB property for the monad $D + (\_)$ is equivalent to the first item. Finally, the second item is equivalent to the equalizer condition of Corollary 2.4, which is equivalent to $\underline{K}$ being fully faithful. ∎

For example, Proposition 2.7 in [14] shows that if $\mathcal{D}$ is extensive then all the monads induced by functors $D/\mathcal{D} \to \mathcal{D}$ are EB. Moreover, the second item of Lemma 5.2 always holds in extensive categories so all objects in an extensive category are detachable.

5.3. Example. If $(\mathcal{D}, \vee, 0)$ is a $\vee$-semilattice, then all the monads induced by $D/\mathcal{D} \to \mathcal{D}$ are idempotent and so, satisfy the Explicit Basis property. But 0 is the only detachable object of $\mathcal{D}$, because $D \vee (\_)$ does not reflect isos if $0 < D$.

For brevity, let us introduce the following terminology.

5.4. DEFINITION. A category with finite coproducts is called *weakly extensive* if all objects are detachable.

In this section we show that modular categories are weakly extensive. But first we discuss the more familiar case of additive categories with kernels.

For the rest of the section assume that $\mathcal{D}$ has finite products and coproducts.

If the unique map $0 \to 1$ from the initial to the terminal object is an iso then $0 : X \to Y$ denotes the *zero map* $X \to 1 = 0 \to Y$.

5.5. DEFINITION. $\mathcal{D}$ is called *linear* if $0 \to 1$ is an iso and for every X, Y in $\mathcal{D}$,

$$X + Y \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}} X \times Y$$

is an iso.

The terminology is taken from [12, 13]. Mitchell calls these categories *semi-additive* (see I.18 in [17]) and Freyd calls them *half-additive* (see 1.591 in [7]). (Co)products in linear categories will be denoted by $\oplus$.

5.6. LEMMA. *Let $\mathcal{D}$ be linear with kernels. For $a : D \to A$ and $s : A \to D \oplus A$ as in Lemma 5.2, the canonical restriction $\overline{s} : A_s \to A$ exists and can be built as the kernel of*

$$A \xrightarrow{s} D \oplus A \xrightarrow{\pi_0} D$$

where $\pi_0 : D \oplus A \to D$ is the projection from the product $D \oplus A$.

PROOF. Let $f : A \to D$ be the composition $\pi_0 s$. The pullback rectangle below

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] & A \arrow[r, "!"] \arrow[d, "in_1"] & 0 \arrow[d, "!"] \\ A \arrow[r, "s"'] & D \oplus A \arrow[r, "\pi_0"'] & D \end{tikzcd}$$

exists by hypothesis and the square on the right is a pullback by the elementary properties of finite products. Then the square on the left is a pullback and it shows that $\ker f$ is the canonical restriction $\overline{s} : A_s \to A$. ∎

Additive categories are typically defined as categories enriched in abelian groups. We will use the following alternative taken from 1.591 in [7].

5.7. DEFINITION. $\mathcal{D}$ is called *additive* if $0 \to 1$ is an iso and for every $f : X \to Y$, the morphism

$$X + Y \xrightarrow{\begin{pmatrix} id & f \\ 0 & id \end{pmatrix}} X \times Y$$

is an iso.

A key result about additive categories with kernels is that the domain of a split epi is canonically iso to the sum of its codomain with its kernel.

5.8. Lemma. *If $\mathcal{D}$ is additive with kernels then it is weakly extensive.*

Proof. For $a : D \to A$ and $s : A \to D \oplus A$ as in Lemma 5.2, a is a section of the map $f = \pi_0 s : A \to D$. By Lemma 5.6 the kernel of f is $\overline{s} : A_s \to A$. By Proposition I.18.5 in [17] the map $[a, \overline{s}] : D \oplus A_s \to A$ is an iso. This proves the first item of Lemma 5.2. To prove the second just observe that in a linear category the following square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\langle 0,id\rangle"'] & Y \arrow[d, "\langle 0,id\rangle"] \\
D \oplus X \arrow[r, "D\oplus g"'] & D \oplus Y
\end{tikzcd}$$

is a pullback for any $g : X \to Y$. ∎

On the other hand, the result above does not hold for arbitrary linear $\mathcal{D}$.

5.9. Example. Consider the linear category $\mathbf{SL}$ of semilattices and the $\vee$-semilattice $A = \{0 < \varepsilon < 1\}$ therein. Let $a : D \to A$ be the sub-semilattice determined by $\{0, \varepsilon\}$. We now show that the monad induced by $D/\mathbf{SL} \to \mathbf{SL}$ is not EB. The semilattice $D \oplus A$ may be pictured as follows:

$$\begin{tikzcd}
 & & (\varepsilon,1) & \\
 & (\varepsilon,\varepsilon) \arrow[ur] & & (0,1) \arrow[ul] \\
(\varepsilon,0) \arrow[ur] & & (0,\varepsilon) \arrow[ul] \arrow[ur] & \\
 & (0,0) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

and the injection $D \to D \oplus A$ maps $x \in D$ to $(x, 0)$ in $D \oplus A$. Consider now the map $s : A \to D \oplus A$ given by $s0 = (0, 0)$, $s\varepsilon = (\varepsilon, 0)$ and $s1 = (\varepsilon, 1)$. Clearly, the triangles below

$$\begin{tikzcd}
 & D \arrow[dl, "a"'] \arrow[d, "in_0"] \arrow[dr, "a"] & \\
A \arrow[r, "s"'] & D \oplus A \arrow[r, "{[a,id]}"'] & A
\end{tikzcd}$$

commute, and the bottom composition $[a, id]s : A \to A$ equals the identity. The canonical restriction $\overline{s} : A_s \to A$ may be calculated as the equalizer of the maps $s, in_1 : A \to D \oplus A$ or as the kernel of $\pi_0 s : A \to D$. Either way, it is easy to see that A_s is initial and so $D \oplus A_s \simeq D$ does not have enough elements to be iso to A.

The same example shows that $D/\mathbf{CMon} \to \mathbf{CMon}$ does not induce an EB monad, where $\mathbf{CMon}$ is the category of commutative monoids. On the other hand, as observed in [3], the fact that retractions are summands also holds in modular categories. So it is not surprising that EB monads also arise in that context.

5.10. Definition. $\mathcal{D}$ satisfies the *modular law* if for every $f : X \to Z$, the canonical map

$$\begin{pmatrix} \langle in_0, f\rangle \\ in_1 \times Z \end{pmatrix} : X + (Y \times Z) \longrightarrow (X+Y) \times Z$$

is an iso for every Y in $\mathcal{D}$.

The modular law plays a key role in Carboni's definition of modular category. But before we recall these, let us observe the following.

5.11. Proposition. *Let $\mathcal{D}$ have finite limits and finite coproducts. If $\mathcal{D}/U$ satisfies the modular law for every U in $\mathcal{D}$ then the monad induced by $D/\mathcal{D} \to \mathcal{D}$ is EB for every D in $\mathcal{D}$.*

Proof. Let $a : D \to A$ and $s : A \to D + A$ be as in Lemma 5.2. Take $U = D + A$ and contemplate the modular law in $\mathcal{D}/(D+A)$ with f being a over $D+A$ as in the diagram on the left below

$$\begin{tikzcd} D \arrow[r, "a"] \arrow[dr, "X=in_0"'] & A \arrow[d, "s=Z"] \\ & D+A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "in_1=Y"] \\ D+A \end{tikzcd}$$

and Y being the injection $in_1 : A \to D + A$ as on the right above. Then $Y \times Z$ is the intersection of the subobjects $in_1 : A \to D + A$ and $s : A \to D + A$. That is, $Y \times Z$ is the composition $A_s \to D + A$ in the equalizer diagram below

$$\begin{tikzcd} A_s \arrow[r, "\overline{s}"] & A \arrow[r, shift left, "in_1"] \arrow[r, shift right, "s"'] & D+A \end{tikzcd}$$

On the other hand, the coproduct $X + Y$ in the slice $\mathcal{D}/(D + A)$ is the terminal object $id : D + A \to D + A$. So $(X + Y) \times Z = Z$. Altogether, the modular law implies that the following map

$$\begin{tikzcd} D+A_s \arrow[rr, "{[a,\overline{s}]}"] \arrow[dr, "{D+\overline{s}=[in_0,in_1\overline{s}]}"'] & & A \arrow[dl, "s"] \\ & D+A & \end{tikzcd}$$

is an iso in $\mathcal{D}/(D + A)$. ∎

We now want to discuss modular categories in the sense of Carboni. From the proof of the main result in [4] one can extract the next one.

5.12. Lemma. *If $\mathcal{D}$ has finite coproducts and finite limits the following are equivalent:*

1. *The diagram*
$$\begin{tikzcd} X \arrow[r, "in_1"] \arrow[d, "f"'] & U+X \arrow[d, "U+f"] \\ U \arrow[r, "in_1"'] & U+U \end{tikzcd}$$
is a pullback for every $f : X \to U$ in $\mathcal{D}$.

2. *The previous condition holds for every $f : X \to 1$.*

3. *The diagram*
$$\begin{tikzcd} X \arrow[r, "in_1"] \arrow[d, "g"'] & U+X \arrow[d, "U+g"] \\ Y \arrow[r, "in_1"'] & U+Y \end{tikzcd}$$
is a pullback for every $g : X \to Y$ and U in $\mathcal{D}$.

Proof. Trivially, the first item implies the second and the third implies the first. To prove that the second item implies the third use the Pasting Lemma to conclude that the square on the left below

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "in_1"'] & Y \arrow[r, "!"] \arrow[d, "in_1"'] & 1 \arrow[d, "in_1"] \\ U+X \arrow[r, "1+g"'] & U+Y \arrow[r, "1+!"'] & 1+1 \end{tikzcd}$$

is a pullback. ∎

5.13. Definition. A category $\mathcal{D}$ with finite limits and coproducts is called *modular* if the following two conditions hold:

1. For every U in $\mathcal{D}$, $\mathcal{D}/U$ satisfies the modular law.

2. The equivalent conditions of Lemma 5.12 hold.

This is, almost exactly, Carboni's definition in [4]. It is clear that modular categories are stable under slicing.

5.14. Example. Additive categories with kernels are modular. See [4] for a proof using embeddings into the category of abelian groups, and Chapter 1 of [18] for an elementary proof. It follows that slices of additive categories are modular and, in fact, the main result in [4] shows that these are all the examples. We sketch a proof in Proposition 7.2 below.

Modular lattices fail to be modular categories because of the second item in Definition 5.13 which, on the other hand, allows to prove the following.

5.15. Corollary. *Modular categories are weakly extensive.*

Proof. By Proposition 5.11 all monads induced by functors $D/\mathcal{D} \to \mathcal{D}$ are EB. The third item in Lemma 5.12 implies that they reflect isos. ■

As observed in [3], modular categories satisfy the much stronger *kernel equivalence*. But notice that this property never holds for non-trivial extensive categories or lattices. The notion of weakly extensive category isolates a non-trivial common aspect of extensivity and modularity. (See Section 3 in [19] for a different but related comparison.)

6. Additive coslices

A basic fact about modularity is that when $\mathcal{D}$ is modular then $1/\mathcal{D}$ is additive and the induced comonad on $1/\mathcal{D}$ is of a very special form as discussed in the present section. The material is known, but not in the form we need it for the results in Section 7. Let $\mathbf{C} = (C, \varepsilon, \delta)$ be a comonad on a category $\mathcal{X}$. Assume that $\mathcal{X}$ has initial object 0 and that coproducts with $C0$ exist in $\mathcal{X}$.

6.1. Definition. The pair $(\mathcal{X}, \mathbf{C})$ satisfies the *coslice condition* if for every $k : X \to CA$ such that the map on the left below

$$C0 + X \xrightarrow{[C!,k]} CA \qquad\qquad X \xrightarrow{k} CA \xrightarrow{\varepsilon} A$$

is an iso, the map on the right above is also an iso.

Let $\mathcal{D}$ be a category with finite coproducts and fix an object D in $\mathcal{D}$. The algebraic functor $D/\mathcal{D} \to \mathcal{D}$ induces a comonad on $D/\mathcal{D}$ that we denote by $\mathbf{C} = (C, \varepsilon, \delta)$. For any object $(A, a : D \to A)$ in the coslice $D/\mathcal{D}$, $C(A, a) = (D + A, in_0)$, and the counit $\varepsilon : (D + A, in_0) \to (A, a)$ is given by $[a, id] : D + A \to A$. The initial object 0 in $D/\mathcal{D}$ is (D, id) and coproducts with $C0 = (D + D, in_0)$ exist because for any (X, x), the coproduct $C0 + (X, x) = (X + D, in_0 x)$ exists, in turn, because the following

$$\begin{tikzcd} D \arrow[r, "x"] \arrow[d, "in_0"'] & X \arrow[d, "in_0"] \\ D + D \arrow[r, "x+D"'] & X + D \end{tikzcd}$$

is a pushout in $\mathcal{D}$.

6.2. Lemma. *The pair $(D/\mathcal{D}, \mathbf{C})$ satisfies the coslice condition.*

Proof. Let $k : (X, x) \to (D + A, in_0)$ be such that

$$C0 + (X, x) = (X + D, in_0 x) \xrightarrow{[k, in_1 a]} (D + A, in_0)$$

is an iso. Its inverse must be of the form $[in_0x, \alpha] : D + A \to X + D$ for some map $\alpha : A \to X + D$. We must show that the maps below

$$X \xrightarrow{k} D + A \xrightarrow{[a,id]} A \qquad\qquad A \xrightarrow{\alpha} X + D \xrightarrow{[id,x]} X$$

are inverse to each other. The next diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, bend right, "in_1"'] & X + D \arrow[r, "{[id,x]}"] \arrow[dr, "{[k,in_1a]}"] & X \arrow[r, "k"] & D + A \arrow[d, "{[a,id]}"] \\
 & D + A \arrow[r, "id"'] \arrow[u, "{[in_0x,\alpha]}"] & D + A \arrow[r, "{[a,id]}"'] & A
\end{tikzcd}$$

proves that the map $A \to X$ is a section of the one $X \to A$. An analogous calculation shows that it is also a retraction. We write down the relevant diagrams for the reader's convenience. First observe that $\alpha a = in_1 : D \to X + D$ as the next diagram

$$\begin{tikzcd}
D \arrow[r, "a"] \arrow[d, "in_1"'] & A \arrow[d, "in_1"] \arrow[dr, "\alpha"] & \\
X + D \arrow[r, "{[k,in_1a]}"'] \arrow[rr, bend right, "id"'] & D + A \arrow[r, "{[in_0x,\alpha]}"'] & X + D
\end{tikzcd}$$

shows. Finally, the diagram below

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[dr, bend right, "in_0"'] & D + A \arrow[r, "{[a,id]}"] \arrow[rr, bend left, "{[in_1,\alpha]}"] \arrow[dr, "{[in_0x,\alpha]}"] & A \arrow[r, "\alpha"] & X + D \arrow[d, "{[id,x]}"] \\
 & X + D \arrow[r, "id"'] \arrow[u, "{[k,in_1a]}"] & X + D \arrow[r, "{[id,x]}"'] & X
\end{tikzcd}$$

proves that the composition $X \to X$ is the identity. ∎

A comonad $\mathbf{C} = (C, \varepsilon, \delta)$ on a category $\mathcal{X}$ with finite products is called *nullary* if the map $\langle C!, \varepsilon\rangle : CA \to C1 \times A$ is an iso for all A in $\mathcal{X}$ [5]. This means that the category of coalgebras is equivalent to $\mathcal{X}/C1$.

6.3. Lemma. *Let* $(\mathcal{X}, \mathbf{C})$ *satisfy the coslice condition. If* $\mathcal{X}$ *is additive with kernels then* $\mathbf{C}$ *is nullary.*

Proof. Let Z be the zero object in $\mathcal{X}$ and denote (co)products by $\oplus$. Also, let us write $z : Z \to A$ and $! : A \to Z$ for the unique maps to and from A. Of course, $Cz : CZ \to CA$ has retraction $C! : CA \to CZ$ and, since $\mathcal{X}$ is additive with kernels, CA is the coproduct of $Cz : CZ \to CA$ with the kernel $k : X \to CA$ of the retraction. In other words,

$[Cz, k] : CZ \oplus X \to CA$ is an iso. Using that k is the kernel of $C! : CA \to CZ$, it is easy to check that the following diagram

$$\begin{tikzcd} CZ \oplus X \arrow[d, "{[Cz,k]}"'] \arrow[dr, "CZ\oplus\varepsilon k"] & \\ CA \arrow[r, "{\langle C!,\varepsilon\rangle}"'] & CZ \oplus A \end{tikzcd}$$

commutes, and the coslice condition implies that $\varepsilon k : X \to A$ is an iso. So both vertical and diagonal maps in the diagram above are isos, and hence, the horizontal map is an iso. ■

Let $\mathcal{D}$ have finite limits and coproducts and let $\mathbf{C}$ be the comonad induced by the adjunction $F \dashv U : 1/\mathcal{D} \to \mathcal{D}$.

6.4. Lemma. *If $1/\mathcal{D}$ is additive then $\mathbf{C}$ coincides with $(F1) \times (\_) : 1/\mathcal{D} \to 1/\mathcal{D}$. Therefore, if it also holds that $F : \mathcal{D} \to 1/\mathcal{D}$ is comonadic then the comparison $\mathcal{D} \to (1/\mathcal{D})/(F1)$ is an equivalence and $\mathcal{D}$ is modular.*

Proof. Finite limits in $\mathcal{D}$ imply that $1/\mathcal{D}$ has kernels. The terminal object 1 in $\mathcal{D}$ coincides with UZ and so, $F1 = CZ$ where $C = FU : 1/\mathcal{D} \to 1/\mathcal{D}$. Finally, if F is comonadic, $\mathcal{D}$ is modular by Example 5.14. ■

Lemma 6.4 is probably well-known because it is related to the notion of essential affineness [3], but it is not explicit there nor in [5].

7. Modularity without descent

The characterization of modular categories proved at the end of [5] rests on Theorem 2.1 loc. cit. which characterizes when the monad determined by the functor $\mathbb{X}^C \to \mathbb{X}/C_0$ is nullary, where C is an internal category in $\mathbb{X}$ and C_0 is its object of objects. We prove an extension of the characterization of modular categories avoiding the descent machinery. At the same time we sketch a proof of Carboni's characterization in [4].

7.1. Lemma. *If $\mathcal{D}$ is modular, $1/\mathcal{D}$ is additive with kernels.*

Proof. The proof in [4] rests on the peculiar properties of the monad on $\mathcal{D}$ induced by the monadic $1/\mathcal{D} \to \mathcal{D}$. First, the Kleisli category is equivalent to the Eilenberg-Moore category because one can apply the modular law to a point $p : 1 \to X$ to conclude that $1 + (0 \times X) \to (1 + 0) \times X = X$ is an iso. Second, every free-algebra has a unique abelian group structure. The proofs in [3] and [19] prove that the 'fibration of points' is additive (reducing the problem to additivity of $\mathcal{D}/0$) and then observe that its fiber over 1 coincides with $1/\mathcal{D}$. ■

Putting together the results in the previous sections we obtain the following extension of the characterizations of modular categories given in [4] and [5].

7.2. Proposition. *Let $\mathcal{D}$ be a category with finite limits and finite coproducts. The following are equivalent.*

1. *$\mathcal{D}$ is modular.*
2. *$1/\mathcal{D}$ is additive and $\mathcal{D} \to 1/\mathcal{D}$ is comonadic.*
3. *$1/\mathcal{D}$ is additive and the monad induced by $1/\mathcal{D} \to \mathcal{D}$ is CEB.*

Moreover, in this case, the canonical $\mathcal{D} \to (1/\mathcal{D})/(1+1, in_0)$ is an equivalence.

Proof. The first item implies the third by Corollary 5.15 and Lemma 7.1. The third item implies the second by Theorem 2.9. Finally, by Lemma 6.4, the second item implies the first and also the last part of the result. ∎

The following corollary (analogous to Theorem 3.4 in [5]) emphasizes the difference between additive and modular categories. Let $\mathbf{Ab}(\mathcal{D})$ denote the category of abelian groups in $\mathcal{D}$ and $\mathbf{Ab}(\mathcal{D}) \to 1/\mathcal{D}$ be the obvious forgetful functor. Also, denote by $\mathbf{M}$ the monad induced by $1/\mathcal{D} \to \mathcal{D}$. We say that the monad is *trivial* if the functor $1/\mathcal{D} \to \mathcal{D}$ is an equivalence.

7.3. Corollary. *If $\mathbf{Ab}(\mathcal{D}) \to 1/\mathcal{D}$ is an equivalence the following hold:*

1. *$\mathcal{D}$ is additive if and only if $\mathbf{M}$ is trivial.*
2. *$\mathcal{D}$ is modular if and only if $\mathbf{M}$ is CEB.*

It is worth mentioning that if $\mathcal{D}$ is modular the left adjoint $\mathcal{D} \to 1/\mathcal{D}$ preserves pullbacks [19]. This connects with Bourn's characterization of modular categories, because pullback-preservation and conservativity imply that *essential affineness* of $1/\mathcal{D}$ is reflected into $\mathcal{D}$. (See Example 2 in Section 4 of [3].)

8. Peano algebras

Let $\mathcal{Y}$ be a Heyting category and $\mathbf{M} = (M, \eta, \mu)$ a monad on it.

8.1. Definition. The *subobject of extreme elements* associated with an $\mathbf{M}$-algebra (A, a) is the subobject of A given by

$$\{x \in A \mid (\forall v \in MA)(av = x \Rightarrow v = \eta x)\} \to A$$

We denote this subobject by $a_\star : \lfloor A, a \rfloor \to A$.

This was introduced in Section 14 in [14] where some examples are discussed. We consider here a different example. Let κ be an object in the topos $\mathbf{Set}/\mathbb{N}$. This object of 'operations indexed by their arity' induces a free algebraic theory (Theorem II.2.1 in [11]) and so, a monad on **Set** that we denote by $\mathbf{M}_\kappa$. One explicit construction of $M_\kappa S$ is given by the set of inductively defined terms using the elements of κ and S. An $\mathbf{M}_\kappa$-algebra (A, a) is determined by a set A together with a family $(f : A^n \to A)_{n \in \mathbb{N}, f \in \kappa n}$, where κn denotes the fiber of κ over $n \in \mathbb{N}$. Each of the functions $f : A^n \to A$ is often called an *operation* of the algebra.

8.2. LEMMA. *For any $\mathbf{M}_\kappa$-algebra (A, a), the subset $\lfloor A, a \rfloor \to A$ is given by the elements of A that are not values of any of the operations of the algebra.*

PROOF. Straightforward. ■

An $\mathbf{M}_\kappa$ algebra (A, a) is called an *induction algebra* if the subset $\lfloor A, a \rfloor \to A$ is generating (p. 503 of [6]). The terminology does not seem appropriate for other monads so let us introduce the following generalization.

8.3. DEFINITION. An $\mathbf{M}$-algebra (A, a) is called *extremally generated* if the composition

$$M\lfloor A, a \rfloor \xrightarrow{Ma_\star} MA \xrightarrow{a} A$$

is regular epi.

The following is a minor rewording of Definition 4.1 loc. cit.

8.4. DEFINITION. An $\mathbf{M}_\kappa$ algebra is called a *Peano algebra* if the following 'generalized Peano axioms' hold:

(P1) all operations are injective,

(P2) the ranges of the operations are pairwise disjoint,

(P3) it is extremally generated.

(Diener actually considers the operations to be *partial* functions, but we will not deal with this sort of generality.)

One fundamental result about Peano algebras is that they are free. The proof in [6] follows as a corollary of Theorem 4.22 there which, in turn, is an application of a careful study of the *algebraic predecessor* relation introduced in Definition 3.1 loc. cit. Explicitly for an $\mathbf{M}_\kappa$-algebra (A, a), and x, y in A, $x <_{(A,a)} y$ if there exists an $f \in \kappa n$ for some n and a n-tuple $\vec{z} \in A^n$ containing x such that $f\vec{z} = y$. The key technical result is that if (A, a) is Peano then the algebraic predecessor relation $<_{(A,a)}$ is well-founded.

On the other hand, free algebraic theories induce EB monads (Section 7 in [14]). So the following alternative proof suggests itself.

8.5. LEMMA. *If (A, a) is Peano then its canonical presentation has a section.*

PROOF. By hypothesis, the composition $(M\lfloor A,a\rfloor,\mu) \to (A,a)$ is the coequalizer of its kernel pair $\pi_0, \pi_1 : K \to (M\lfloor A,a\rfloor,\mu)$. So it is enough to prove that the 'twisted fork' below

$$\begin{tikzcd}
K \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & (M\lfloor A,a\rfloor,\mu) \arrow[r, "Ma_\star"] \arrow[drr, "Ma_\star"'] & (MA,\mu) \arrow[r, "a"] & (A,a) \arrow[d, dotted, "s"] \\
 & & & (MA,\mu)
\end{tikzcd}$$

commutes, inducing the indicated map $s : (A,a) \to (MA,\mu)$. In order to do this, let $f[\vec{x}], g[\vec{y}] \in M\lfloor A,a\rfloor$ be such that $f(\vec{x}) = g(\vec{y})$ in A. We show, by induction on the number of symbols forming $f[\vec{x}]$ and $g[\vec{y}]$, that $f[\vec{x}] = g[\vec{y}] \in MA$. First assume that $f[\vec{x}] = \eta x \in M\lfloor A,a\rfloor$ for some $x \in \lfloor A,a\rfloor$. Then $x = g(\vec{y}) \in A$ and since x is extreme, $g[\vec{y}] = \eta x$; so $f[\vec{x}] = g[\vec{y}] \in MA$. Similarly if $g[\vec{y}] = \eta y$. This completes the base case. Now assume that $f[\vec{x}]$ and $g[\vec{y}]$ are non-trivial. That is, $f[\vec{x}] = f_0(p_1[\vec{x}], \ldots, p_m[\vec{x}])$ and $g[\vec{y}] = g_0(q_1[\vec{y}], \ldots, q_n[\vec{y}])$ for some 'operations' $f_0 \in \kappa m$ and $g_0 \in \kappa n$. Then, by hypothesis, $f(\vec{x}) = f_0(p_1\vec{x}, \ldots, p_m\vec{x})$ and $g(\vec{y}) = g_0(q_1\vec{y}, \ldots, q_n\vec{y})$ in A. Since operations are disjoint, $f_0 = g_0$ and $m = n$. As operations are injective $p_i\vec{x} = q_i\vec{y} \in A$ for every $1 \le i \le m$. So $p_i[\vec{x}] = q_i[\vec{y}] \in M\lfloor A,a\rfloor$ by inductive hypothesis and the proof is complete. ∎

The rest follows from the elementary theory of EB monads in Heyting categories.

8.6. PROPOSITION. *Every Peano algebra is free on its subset of extreme elements. More explicitly: if (A,a) is a Peano algebra then the canonical map $M\lfloor A,a\rfloor \to A$ is an iso.*

PROOF. It is easy to check that for every set S, $\lfloor MS,\mu\rfloor \to MS$ coincides with the unit $\eta : S \to MS$. So Proposition 14.2 in [14] is applicable and implies that the canonical restriction of the section $s : (A,a) \to (MA,\mu)$ built in Lemma 8.5 coincides with $\lfloor A,a\rfloor \to A$. ∎

Lawvere's construction of the algebraic theory associated to an algebraic category makes it evident that the properties of free algebras are closely related to properties of the theory. The work on Peano algebras suggests a specific condition that is closely related to the Explicit Basis property. Fix a monad $\mathbf{M}$ on a category $\mathcal{Y}$.

8.7. DEFINITION. An $\mathbf{M}$-algebra (A,a) is said to have *monic operations* if the canonical factorization

$$\begin{tikzcd}
MA \arrow[d, dotted] \arrow[dr, "\Delta"] & & & \\
E \arrow[r] & M(A\times A) \arrow[r, shift left, "M\pi_0"] \arrow[r, shift right, "M\pi_1"'] & MA \arrow[r, "a"] & A
\end{tikzcd}$$

of $M\Delta : MA \to M(A \times A)$ via the equalizer $E \to M(A \times A)$ is an iso.

The following simple result gives a more external feeling for the definition.

8.8. LEMMA. *Let (A, a) have monic operations. If the diagram on the left below commutes*

$$X \xrightarrow{x} MU \underset{Mg}{\overset{Mf}{\rightrightarrows}} MA \xrightarrow{a} A \qquad\qquad X \xrightarrow{x} MU \underset{Mg}{\overset{Mf}{\rightrightarrows}} MA$$

then so does the diagram on the right.

For example, if (A, a) is an algebra for $\mathbf{M}_\kappa$ induced by $\kappa \in \mathbf{Set}/\mathbb{N}$ as discussed above then it has monic operations in the sense of Definition 8.7 if and only if for every $f \in \kappa n$, the function $f : A^n \to A$ is injective.

8.9. PROPOSITION. *Let $\mathcal{Y}$ have finite limits, $L \dashv R : \mathcal{X} \to \mathcal{Y}$ be a bimonadic adjunction and $\mathbf{M}$ be the induced monad. If every free $\mathbf{M}$-algebra has monic operations then $\mathbf{M}$ is EB.*

PROOF. If $s : (A, a) \to (MA, \mu)$ is a section for the canonical presentation of (A, a) then the diagram on the left below

$$A \xrightarrow{s} MA \underset{Ms}{\overset{M\eta}{\rightrightarrows}} MMA \xrightarrow{\mu} MA \qquad\qquad A \xrightarrow{s} MA \underset{Ms}{\overset{M\eta}{\rightrightarrows}} MMA$$

commutes. So the diagram on the right commutes by Lemma 8.8. In other words, Redundant Coassociativity holds. ■

This gives a different explanation for why free theories are EB. The simplest non-free example is probably the theory presented with two unary operations f and g subject to the equation $fg = gf$. Another simple example is the theory presented with two unary operations f and g and a constant c such that $fc = gc$. The characterization of the algebraic theories that induce EB monads remains an open problem.

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