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# FUNCTORIAL AND ALGEBRAIC PROPERTIES OF BROWN'S $\mathcal{P}$ FUNCTOR

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Abstract. In 1975 E. M. Brown constructed a functor $\mathcal{P}$ which carries the tower of fundamental groups of the end of a (nice) space to the Brown-Grossman fundamental group. In this work, we study this functor and its extensions and analogues defined for pro-sets, pro-pointed sets, pro-groups and pro-abelian groups. The new versions of the $\mathcal{P}$ functor are provided with more algebraic structure. Examples given in the paper prove that in general the $\mathcal{P}$ functors are not faithful, however, one of our main results establishes that the restrictions of the corresponding $\mathcal{P}$ functors to the full subcategories of towers are faithful. We also prove that the restrictions of the $\mathcal{P}$ functors to the corresponding full subcategories of finitely generated towers are also full. Consequently, in these cases, the towers of objects in the categories of sets, pointed sets, groups and abelian groups, can be replaced by adequate algebraic models ($M$-sets, $M$-pointed sets, near-modules and modules.) The article also contains the construction of left adjoints for the $\mathcal{P}$ functors.

## 1. Introduction

This article contains a detailed study of the properties of the $\mathcal{P}$ functor. An interesting consequence is that in many cases we can replace an inverse system of sets or groups by special algebraic models that contain all the information of the corresponding inverse systems. Firstly we recall the context in which L. R. Taylor and E.M. Brown defined the $\Delta$-homotopy groups and the $\mathcal{P}$ functor.

In 1971, L.R. Taylor [Tay] defined the fundamental $\Delta$-group of a space by taking a set of base points such that any infinite path-component of the complement of a compact subspace contains base points. He used the fundamental groups of these path-components based at these base points to define the fundamental $\Delta$-group of a space.

In 1975, E.M. Brown [Br] defined the proper fundamental group of a space with a base ray by using the string of 1–spheres, ${}^BS^1$, which is defined by attaching one 1–sphere $S^1$ at each positive integer of the half line $[0, +\infty)$. Given a space $X$ with a base ray, he considered the proper fundamental group $\pi_1^B(X) = [{}^BS^1, X]_\infty$ as the set of germs at infinity of proper maps from ${}^BS^1$ to $X$, modulo germs at infinity of proper homotopies. Let $X$ be a well rayed space (the inclusion of the base ray is a cofibration) and suppose that $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots$, is a cofinal sequence of compact subsets. Denote by $X_i = cl(X - K_i)/\text{ray}$ and by $\varepsilon X = \{X_i\}$ the associated end tower of $X$. The

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fundamental pro-group $\pi_1\varepsilon X$ of $X$ is isomorphic to the tower $\{\pi_1 X_i\}$. Brown defined a functor $\mathcal{P}\colon towGps \longrightarrow Gps$ satisfying $\mathcal{P}\pi_1\varepsilon X \cong \pi_1^B(X)$. If one takes a one ended space having a countable base of neighbourhoods at infinity, a base ray gives a set of base points and it can be checked that the fundamental $\Delta$-group of this space agrees with the global fundamental Brown group, which is defined by considering global proper maps and global proper homotopies instead of proper germs, see [H-P].

Later Grossman [Gr.1, Gr.2, Gr.3] developed a homotopy theory for towers of simplicial sets and defined the analogues of Brown's groups for towers of simplicial sets. Using a well known exact sequence, he proved that his notion of fundamental group, $\pi_1^G(Y)$, was isomorphic to $towGps(c\mathbf{Z}, \{\pi_1 Y_i\})$, where $c\mathbf{Z}$ is defined by $(c\mathbf{Z})_i = \sum_{j\geq i} \mathbf{Z}_j$, where $\mathbf{Z}_j \cong \mathbf{Z}$ and $\Sigma$ denotes the coproduct in the category of groups. Applying the Edwards-Hastings embedding theorem [E-H] we have that $\pi_1^B(X) \cong \pi_1^G(\varepsilon X)$. Therefore the Edwards-Hastings embedding relates the Brown definition of the $\mathcal{P}$ functor and the functor $towGps(c\mathbf{Z}, -)$. It is not hard to see that $\mathcal{P}$ and $towGps(c\mathbf{Z}, -)$ are isomorphic for any tower of groups and then the $\mathcal{P}$ functor can be seen as a representable hom-group functor on the category $towGps$.

We have considered this last formulation of the $\mathcal{P}$ functor in order to define our more algebraically structured version of the $\mathcal{P}$ functor. Next we give some notation and our definition of the $\mathcal{P}$ functor and afterwards we establish some of the main results of this work.

Given an object $H$ of a category $\mathcal{C}$, we can consider the class of objects generated from it by taking arbitrary sums of copies of $H$ and effective epimorphisms. When this class contains all the objects of the category it is said that $H$ is a generator for the category $\mathcal{C}$. Notice that the one-point set $*$ is a generator for the category $Set$ of sets, the two-point set $S^0$ is a generator for the category $Set_*$ of pointed sets, the infinite cyclic group $\mathbf{Z}$ is a generator for the category $Grp$ of groups and the infinite cyclic abelian group, denoted in this paper by $\mathbf{Z}_a$, is a generator for the category $Ab$ of abelian groups. We will denote by $C$ one of the categories: $Set,\ Set_*,\ Grp, Ab$. The generator of $C$ will be denoted by $G$.

Associated with the category $C$, one has the category, $towC$, of towers in $C$ and the category, $proC$, of pro-objects in $C$. The object $G$ induces a pro-object $cG\colon \mathbf{N} \longrightarrow C$ defined by

$$(cG)_i = \sum_{j\geq i} G, \quad i \in \mathbf{N}.$$

Given a pro-object $X$ in $proC$, one has the canonical action

$$proC(cG, X) \times proC(cG, cG) \longrightarrow proC(cG, X) : (f, \varphi) \to f\varphi.$$

For the different cases $C = Set,\ Set_*,\ Grp,\ Ab$, we note that $proC(cG, X)$ is a set, a pointed set, a group and an abelian group, respectively. On the other hand, $\mathcal{P}cG = proC(cG, cG)$ has respectively the structure of a monoid, a 0-monoid (see section 3), a near-ring and a ring. As a consequence of this fact, we will use different algebraic categories, but because they have many common functorial properties we will use the following unified notation:

1) If $C = Set$ and $G = *$, $C_{\mathcal{P}cG}$ will be the category of $\mathcal{P}cG$-sets. A $\mathcal{P}cG$-set consists of a set $X$ together with an action of the monoid $\mathcal{P}cG = proC(cG, cG)$.

2) If $C = Set_*$ and $G = S^0$, $C_{\mathcal{P}cG}$ will be the category of $\mathcal{P}cG$-pointed sets. In this case, the structure is given by an action of a 0-monoid on a pointed set.

3) If $C = Grp$ and $G = \mathbf{Z}$, $C_{\mathcal{P}cG}$ will be the category of $\mathcal{P}cG$-groups. Now the structure is given by an action of a near-ring on a group, see sections 3 and 4.

4) If $C = Ab$ and $G = \mathbf{Z}_a$, $C_{\mathcal{P}cG}$ will be the category of $\mathcal{P}cG$-abelian groups (modules over the unitary ring $\mathcal{P}cG$).

Using this notation, given a pro-object $X$ in $proC$, $\mathcal{P}X = proC(cG, X)$ provided with the action of $proC(cG, cG)$ determines an object of the category $C_{\mathcal{P}cG}$. This defines the functor $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$.

Next we introduce some of the main results of the paper:

THEOREM 4.4 *$\mathcal{P}\colon towC \longrightarrow C_{\mathcal{P}cG}$ is a faithful functor.*

This establishes that the restriction of the $\mathcal{P}$ functor to the full subcategory of towers in $C$ is a faithful functor. It is interesting to note that the extended $\mathcal{P}$ functor, for instance from pro-abelian groups to modules is not faithful, see Corollary 7.16.

Another important result of the paper states that the restriction of the $\mathcal{P}$ functor to finitely generated towers is also full.

THEOREM 4.11 *Let $X$ be an object in $towC$. If $X$ is finitely generated, then $X$ is admissible in $towC$. Consequently, the restriction functor $\mathcal{P}\colon towC/fg \longrightarrow C_{\mathcal{P}cG}$ is a full embedding, where $towC/fg$ denotes the full subcategory of $towC$ determined by finitely generated towers.*

For finitely generated towers, we are able to replace a tower of objects by a single object with some additional algebraic structure. In this paper we have only considered this kind of construction for towers of sets and towers of groups, however, many of the poofs are established by using very general functorial methods. Therefore part of the constructions and results can be extended to towers and pro-objects in more general categories.

For the case $C = Grp$, the main results of Chipman [Ch.1, Ch.2] stated for towers of finitely generated groups, are obtained from Theorem 4.11 as corollaries. We point out that the class of finitely generated towers of groups is larger than the class of towers of finitely generated groups.

Fortunately, some very important examples of towers of sets are finitely generated, for example, the tower of $\pi_0$'s of the end of a locally compact, $\sigma$-compact Hausdorff space or the tower of $\pi_0$'s obtained by the Čech nerve for a compact metric space. In these cases, it is easy to prove that the fundamental pro-groups are finitely generated and therefore the $\mathcal{P}$ functor will work nicely on this kind of pro-group. In the abelian case the towers of singular homology groups coming from proper homotopy and shape theory are finitely generated. However, we do not know if for towers of higher homotopy groups, the abelian version of the $\mathcal{P}$ functor is going to be a full embedding.

As a consequence of Theorem 4.11, we get a full embedding of the category of zero-dimensional compact metrisable spaces and continuous maps into the algebraic category of $\mathcal{P}c*$-sets. We also obtain similar embeddings for the corresponding categories of topological groups and topological abelian groups.

In this work, we also give the construction of left adjoints for the $\mathcal{P}$ functors.

THEOREM 6.3 *The functor* $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ *has a left adjoint* $\mathcal{L}\colon C_{\mathcal{P}cG} \longrightarrow proC$.

This left adjoint, for instance for the abelian case, constructs the pro-abelian group associated with a module over the ring $\mathcal{P}c\mathbf{Z}_a$. It is interesting to note that $\mathcal{P}c\mathbf{Z}_a$ is isomorphic to the ring of locally finite matrices modulo finite matrices which was used by Farrell-Wagoner [F-W.1, F-W.2] to define the Whitehead torsion of an infinite complex.

Next we include some additional remarks about other results and constructions developed in this paper.

In section 2, we analyse some nice properties of categories of the form $proC$. Given a strongly cofinite set $I$, we prove in Theorem 2.4 that the full subcategory $pro_IC$, determined by the objects of $proC$ indexed by $I$, is equivalent to a category of right fractions $C^I\Sigma^{-1}$ in the sense of Gabriel and Zisman [G–Z]. When the directed set of natural numbers $I = \mathbf{N}$ is considered, the category $pro_{\mathbf{N}}C$ is usually denoted by $towC$. From Theorem 2.4 we have that $towC$ can be obtained as a category of right fractions of $C^{\mathbf{N}}$, however, for this case we also prove in Theorem 2.9 that the category $towC$ is equivalent to a category of left fractions $\Gamma^{-1}C^{\mathbf{N}}$. This fact has the following nice consequence: The hom-set $towC(cG, X)$ can be expressed as a colimit and this gives the definition of the $\mathcal{P}$ functor given by Brown or if we use the standard definition given as a limit of colimits, we nearly obtain the definition of $\Delta$-homotopy group "at infinity" given by Taylor.

When we are working with categories of sets, pointed sets, groups and abelian groups we usually consider free, forgetful and abelianization functors. For the corresponding "towcategories" and "procategories", we also have analogous induced functors. The different versions of $\mathcal{P}$ are functors into categories of $\mathcal{P}c*$-sets, $\mathcal{P}cS^0$- pointed sets, $\mathcal{P}c\mathbf{Z}$-groups and $\mathcal{P}c\mathbf{Z}_a$-abelian groups. For these categories, we analyse, in section 5, the definition and properties of the analogues of this kind of functor. For example, the left adjoint of the natural inclusion of the category of $\mathcal{P}c\mathbf{Z}_a$-abelian groups into the category of $\mathcal{P}c\mathbf{Z}$-groups is a kind of "distributivization" functor. Given a $\mathcal{P}c\mathbf{Z}$-group $X$, a quotient $dX$ is defined by considering the relations which are necessary to obtain a right distributive action.

An important result of section 5 is Theorem 5.8. In terms of proper homotopy, Theorem 5.8 proves that the $\mathcal{P}$ functor sends the abelianization of the tower of fundamental groups to the "distributivization" of the fundamental Brown-Grossman group. It is not hard to find topological examples where the abelianization of the fundamental Brown-Grossman group produces a type of first homology group, which is not naturally isomorphic to the "distributivization" of the Brown-Grossman group.

Finally, we have developed section 7 to solve some of the theoretical questions that have arisen from writing the paper. We see that the full subcategory of locally structured topological abelian groups admits a full embedding into the category of global pro-objects of abelian groups. As a consequence of the relations between these categories we obtain Corollaries 7.10 and 7.13, which are the main results of the section. In these corollaries it is proved that neither $(towAb, Ab)$ nor $towAb$ have countable sums, and therefore, neither

$(towAb, Ab)$ nor $towAb$ are equivalent to a category of modules.

## 2. Procategories and categories of fractions

The category $proC$, where $C$ is a given category, was introduced by A. Grothendieck [Gro]. A study of some properties of this category can be seen in the appendix of [A–M], the monograph of [E–H] or in the books of [M–S] and [C-P].

The objects of $proC$ are functors $X: I \to C$, where $I$ is a small left filtering category, and the set of morphisms from $X: I \to C$ to $Y: J \to C$ is defined by the formula

$$proC(X, Y) = \lim_j \operatorname{colim}_i C(X_i, Y_j).$$

A morphism from $X$ to $Y$ can be represented by $(\varphi, \{f_j\})$ where $\varphi: J \to I$ is a map and $f_j: X_{\varphi j} \to Y_j$ is a morphism of $C$ such that if $j \to j'$ is a morphism of $J$, there are $i \in I$ and morphisms $i \to \varphi j$, $i \to \varphi j'$ such that the composite $X_i \to X_{\varphi j} \to Y_j \to Y_{j'}$ is equal to the composite $X_i \to X_{\varphi j'} \to Y_{j'}$.

Notice that if $I$ is a strongly directed set and $J$ is just a set, then $Maps(J, I)$ is also a directed set. It is easy to see that if $I$ is a strongly directed set and $J$ is a strongly cofinite directed set, then the natural inclusion $I^J \to Maps(J, I)$ is a cofinal functor, see [M–S, page 9], where $I^J$ denotes the strongly directed set of functors from $J$ to $I$. As a consequence of this fact, if $I, J$ are strongly cofinite directed sets, any morphism of $proC$ from $X$ to $Y$ can be represented by $(\varphi, f)$ where $\varphi \in I^J$ and $f_j: (X\varphi)_j \to Y_j$ is a level morphism.

Let $C^{scd}$ denote the category whose objects are functors $X: I \to C$, where $I$ is a strongly cofinite directed set, and a morphism from $X: I \to C$ to $Y: J \to C$ is given by a functor $\varphi: J \to I$ and by a natural transformation $f: X\varphi \to Y$, where $X\varphi$ is the composition of the functors $\varphi$ and $X$. Given a strongly cofinite directed set $I$, we also consider the subcategory $C^I$ of $C^{scd}$ given by objects indexed by $I$ and morphisms of the form $(\varphi, f)$ where $\varphi = id_I$.

If $J$, $I$ are strongly cofinite directed sets, then $I^J$ the set of functors from $J$ to $I$ is a strongly directed set ($\varphi, \psi \in I^J, \varphi \geq \psi$ if $\varphi(j) \geq \psi(j), j \in J$) and can be considered as a category. The evaluation functor $e: C^I \times I^J \to C^J$ is defined by $e(x, \varphi) = X\varphi = X_*\varphi$. A fixed $\varphi$ induces a functor $-_*\varphi: C^I \to C^J$, which sends $f: X \to Y$ to $f_*\varphi: X_*\varphi \longrightarrow Y_*\varphi$, and a fixed $X$ induces a functor $X_*-: I^J \to C^J$ sending $\varphi \geq \psi$ to $X_*{}^{\varphi}_{\psi}: X_*\varphi \longrightarrow X_*\psi$.

Let $pro_{scd}C$ denote the full subcategory of $proC$ defined by the objects of $proC$ indexed by strongly cofinite directed sets. If $X: I \to C$ and $Y: J \to C$ are objects in $pro_{scd}C$, we can take into account that $I^J$ is cofinal in $Maps(J, I)$ to see that:

$$pro_{scd}C(X, Y) = proC(X, Y) \cong \operatorname{colim}_{\varphi \in I^J} C^J(X_*\varphi, Y)$$

That is, $proC(X, Y)$ is the colimit of the functor

$$(I^J)^{op} \xrightarrow{X_*-} (C^J)^{op} \xrightarrow{C^J(-,Y)} Set$$

Edwards and Hastings [E–H] gave the construction (the Mardešić trick, see also [M–S], page 15]) of a functor $M: proC \longrightarrow pro_{scd}C$ that together with the inclusion $pro_{scd}C \longrightarrow$ $\longrightarrow proC$ give an equivalence of categories. We note that the category $pro_{scd}C$ is a quotient of the category $C^{scd}$; that is, $pro_{scd}C(X,Y)$ is a quotient of $C^{scd}(X,Y)$.

We include the following result about cofinal subsets of $I^I$.

2.1. Lemma. *Let $I$ be a strongly cofinite directed set and consider $I^I_{id} = \{\varphi \in I^I \mid \varphi \geq id\}$, then*

*1) the inclusion $I^I_{id} \longrightarrow I^I$ is cofinal,*

*2) for the case $I = \mathbf{N}$ the directed set of non-negative integers, if $In(\mathbf{N}^{\mathbf{N}}) = \{\varphi \in \mathbf{N}^{\mathbf{N}} \mid \varphi$ is injective $\}$ and $In(\mathbf{N}^{\mathbf{N}}_{id}) = \mathbf{N}^{\mathbf{N}}_{id} \cap In(\mathbf{N}^{\mathbf{N}})$, then $In(\mathbf{N}^{\mathbf{N}}) \longrightarrow \mathbf{N}^{\mathbf{N}}$ and $In(\mathbf{N}^{\mathbf{N}}_{id}) \longrightarrow \mathbf{N}^{\mathbf{N}}$ are cofinal.*

Next, for a given strongly cofinite directed set $I$, we analyse the relationship between $C^I$ and $pro_IC$, the full subcategory of $pro_{scd}C$ defined by the objects indexed by a fixed $I$. We are going to see that $pro_IC$ is a category of right fractions of $C^I$. For this purpose, first we recall, see [G–Z], under which conditions a class $\Sigma$ of morphisms of a category $\mathcal{C}$ admits a calculus of left (or right) fractions.

A class $\Sigma$ of morphisms of $\mathcal{C}$ admits a calculus of left fractions if $\Sigma$ satisfies the following properties:

a) The identities of $\mathcal{C}$ are in $\Sigma$.
b) If $u: X \to Y$ and $v: Y \to Z$ are in $\Sigma$, their composition $vu: X \to Z$ is in $\Sigma$.
c) For each diagram $X' \xleftarrow{s} X \xrightarrow{u} Y$ where $s$ is in $\Sigma$, there is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X' & \xrightarrow[u']{} & Y' \end{array}$$

where $t$ is in $\Sigma$.
d) If $f, g: X \to Y$ are morphisms of $\mathcal{C}$ and if $s: X' \to X$ is a morphisms of $\Sigma$ such that $fs = gs$, there exists a morphism $t: Y \to Y'$ of $\Sigma$ such that $tf = tg$.

If we replace the conditions c) and d) by the conditions c′) and d′) below, the class $\Sigma$ is said to admit a calculus of right fractions.

c′) For each diagram $X' \xrightarrow{u'} Y' \xleftarrow{t} Y$ where $t$ is in $\Sigma$, there is a diagram $X' \xleftarrow{s} X \xrightarrow{u} Y$ such that $u's = tu$ and $s$ is in $\Sigma$.
d′) If $f, g: X \to Y$ are morphisms of $\mathcal{C}$ and if $t: Y \to Y'$ is a morphism of $\Sigma$ such that $tf = tg$, there exists a morphism $s: X' \to X$ of $\Sigma$ such that $fs = gs$.

Now for a given strongly cofinite directed set $I$, consider the class $\Sigma$ of $C^I$ defined by the morphisms of the form $X_{*}{}^{\varphi}_{id}: X_*\varphi \to X$ where $\varphi \in I^I_{id}$.

2.2. PROPOSITION. *$\Sigma$ admits a calculus of right fractions.*

PROOF. It is clear that $X_{*id}^{\;id} = id_X$ and that $(X_{*id}^{\;\varphi})((X_*\varphi)_{*id}^{\;\psi}) = X_{*id}^{\;\varphi\psi}$. These imply a) and b). By considering the evaluation functor $C^I \times I_{id}^I \to C^I$, $(X,\varphi) \to X_*\varphi$, we obtain the commutative square

$$\begin{tikzcd} X_*\varphi \arrow[r, "f_*\varphi"] \arrow[d, "X_{*id}^{\;\varphi}"'] & Y_*\varphi \arrow[d, "Y_{*id}^{\;\varphi}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Hence we have that c′) is satisfied.

Finally, if $f, g\colon X \to Y_*\varphi$ are morphisms such that $(Y_{*id}^{\;\varphi})f = (Y_{*id}^{\;\varphi})g$, we have that
$((Y_*\varphi)_{*id}^{\;\varphi})(f_*\varphi) = ((Y_{*id}^{\;\varphi})_*\varphi)(f_*\varphi) = ((Y_{*id}^{\;\varphi})_*\varphi)(g_*\varphi) = ((Y_*\varphi)_{*id}^{\;\varphi})(g_*\varphi)$,
$((Y_*\varphi)_{*id}^{\;\varphi})(f_*\varphi) = f(X_{*id}^{\;\varphi})$,
$((Y_*\varphi)_{*id}^{\;\varphi})(g_*\varphi) = g(X_{*id}^{\;\varphi})$.
Therefore it follows that $f(X_{*id}^{\;\varphi}) = g(X_{*id}^{\;\varphi})$, and so d′) is also satisfied. ∎

Given a strongly cofinite directed set $I$, we denote by $C^{scd_I}$ and $pro_I C$ the full subcategories of $C^{scd}$ and $proC$, respectively, determined by objects indexed by $I$. Now to compare $C^I\Sigma^{-1}$ and $pro_I C$, we consider the diagram

$$\begin{tikzcd} C^I \arrow[r] \arrow[d] & C^{scd_I} \arrow[d] \\ C^I\Sigma^{-1} & pro_I C \end{tikzcd}$$

In order to have an induced functor, it suffices to see that a morphism of $\Sigma$ is sent to an isomorphism of $pro_I C$. A morphism of $\Sigma$ is of the form $X_{*id}^{\;\varphi}\colon X_*\varphi \to X$, $\varphi \in I_{id}^I$. We also consider the morphism $\sigma_\varphi^{id}(X) = (\varphi, id_{X_*\varphi})\colon X \to X_*\varphi$ in the category $C^{scd_I}$. Using this notation we have:

2.3. LEMMA. *The morphisms $X_{*id}^{\;\varphi}$, $\sigma_\varphi^{id}(X)$ induce an isomorphism in $pro_I C$.*

PROOF. Consider the composites

$$X_*\varphi \xrightarrow{(id_I, X_{*id}^{\;\varphi})} X \xrightarrow{(\varphi, id_{X_*\varphi})} X_*\varphi$$

$$X \xrightarrow{(\varphi, id_{X_*\varphi})} X_*\varphi \xrightarrow{(id_I, X_{*id}^{\;\varphi})} X.$$

We have that

$$(\varphi, id_{X_*\varphi})(id_I, X_*{}^{\varphi}_{id}) = (\varphi, (X_*{}^{\varphi}_{id})_*\varphi).$$

Because the following diagram is commutative

$$\begin{tikzcd}
(X_*\varphi)_*\varphi \arrow[r, "(X_*\varphi)_*{}^{\varphi}_{\varphi}"] \arrow[d, "(X_*\varphi)_*{}^{\varphi}_{id}"'] & (X_*\varphi)_*\varphi \arrow[d, "(X_*{}^{\varphi}_{id})_*\varphi"] \\
X_*\varphi \arrow[r, "id_{X_*\varphi}"'] & X_*\varphi,
\end{tikzcd}$$

it follows that $(id_I, id_{X_*\varphi}) = (\varphi, (X_*{}^{\varphi}_{id})_*\varphi)$ in $pro_I C$.

On the other hand, we have

$$(id_I, X_*{}^{\varphi}_{id})(\varphi, id_{X_*\varphi}) = (\varphi, X_*{}^{\varphi}_{id}).$$

Since the diagram

$$\begin{tikzcd}
X_*\varphi \arrow[r, "X_*{}^{\varphi}_{\varphi}"] \arrow[d, "X_*{}^{\varphi}_{id}"'] & X_*\varphi \arrow[d, "X_*{}^{\varphi}_{id}"] \\
X \arrow[r, "id_X"'] & X
\end{tikzcd}$$

is commutative, we have that $(\varphi, X_*{}^{\varphi}_{id}) = (id_I, id_X)$ in $pro_I C$. ∎

2.4. Theorem. *The induced functor $C^I\ \Sigma^{-1} \longrightarrow pro_I C$ is an equivalence of categories.*

Proof. Since $I^I_{id}$ is cofinal in $I^I$, by Lemma 2.1, a morphism $X \to Y$ in $pro_I C$ can be represented in $C^{scd_I}$ by

$$X \xrightarrow{\sigma^{id}_{\varphi}(X)} X_*\varphi \xrightarrow{f} Y$$

where $f$ is in $C^I$. By Lemma 2.3, in the category $pro_I C$ we have that $f\ \sigma^{id}_{\varphi}(X) = f(X_*{}^{\varphi}_{id})^{-1}$. Therefore $C^I\Sigma^{-1}(X,Y) \to pro_I C(X,Y)$ is surjective.

Given two morphisms $X \xleftarrow{X_*{}^{\varphi}_{id}} X_*\varphi \xrightarrow{f} Y$, $X \xleftarrow{X_*{}^{\psi}_{id}} X_*\psi \xrightarrow{g} Y$ in

$C^I\Sigma^{-1}$ such that $(\varphi, f) = (\psi, g)$ in $pro_I C$, there is a $\theta \in I^I_{id}$ such that the diagram

$$\begin{tikzcd}
 & X_*\varphi \arrow[dl] \arrow[dr, "f"] & \\
X & X_*\theta \arrow[l] \arrow[u, "X_{*\varphi}^{\theta}"'] \arrow[d, "X_{*\psi}^{\theta}"] & Y \\
 & X_*\psi \arrow[ul] \arrow[ur, "g"'] &
\end{tikzcd}$$

is commutative in $C^I$. Therefore in $C^I\Sigma^{-1}$ we have that

$$f(X_{*id}^{\varphi})^{-1} = f(X_{*\varphi}^{\theta})(X_{*id}^{\theta})^{-1} = g(X_{*\psi}^{\theta})(X_{*id}^{\theta})^{-1} = g(X_{*id}^{\psi})^{-1}.$$

Hence $C^I\Sigma^{-1}(X, Y) \cong pro_I C(X, Y)$.

■

Next, we study a particular but important case. We consider the strongly cofinite directed set $\mathbf{N}$ of natural numbers. In this case, the category $pro_{\mathbf{N}}C$ is usually denoted by $towC$. We are going to prove that if $C$ has a final object $*$, the category $towC$ can also be obtained from $C^{\mathbf{N}}$ as a category of left fractions. As before $\mathbf{N}^{\mathbf{N}}$ denotes the strongly directed set of functors from $\mathbf{N}$ to $\mathbf{N}$. We will use the following notation $\mathbf{N}^+ = \{-1\} \cup \mathbf{N}$, $(\mathbf{N}^+)^+ = \mathbf{N}^{++} = \{-2\} \cup \mathbf{N}^+$ and $in\colon \mathbf{N} \to \mathbf{N}^+$ denotes the inclusion. We also have the strongly directed sets:

$$\begin{aligned}
Cof(\mathbf{N}^{\mathbf{N}}) \quad &= \{\varphi \in \mathbf{N}^{\mathbf{N}} \mid \varphi \text{ is cofinal}\} \\
Cof((\mathbf{N}^+)^{\mathbf{N}}) \quad &= \{\psi \in (\mathbf{N}^+)^{\mathbf{N}} \mid \psi \text{ is cofinal}\}.
\end{aligned}$$

Notice that we have the relations:

$$In(\mathbf{N}^{\mathbf{N}}_{id}) \subset In(\mathbf{N}^{\mathbf{N}}) \subset Cof(\mathbf{N}^{\mathbf{N}}) \subset \mathbf{N}^{\mathbf{N}}.$$

Given a $\varphi \in Cof(\mathbf{N}^{\mathbf{N}})$, define $\bar{\varphi}\colon \mathbf{N} \to \mathbf{N}^+$ as follows: If $n < \varphi(0)$, then $\bar{\varphi}(n) = -1$. Otherwise, there is an $i \in \mathbf{N}$ such that $\varphi(i) \leq n < \varphi(i+1)$, and then define $\bar{\varphi}(n) = i$. Suppose that $n \leq m$, we have that if $n < \varphi(0)$, then $\bar{\varphi}(n) \leq \bar{\varphi}(m)$, otherwise $\varphi(i) \leq n < \varphi(i+1)$, $n \leq m$, and we again have that $\bar{\varphi}(n) \leq \bar{\varphi}(m)$. Similarly, it is easy to check that $\overline{id}_{\mathbf{N}} = in$ and that if $\varphi \leq \psi$, then $\bar{\varphi} \geq \bar{\psi}$. Therefore we have defined a contravariant functor

$$^{-}\colon Cof(\mathbf{N}^{\mathbf{N}}) \longrightarrow Cof((\mathbf{N}^+)^{\mathbf{N}}).$$

If $C$ is a category with a final object $*$, we also consider the functor $^+ \colon C^{\mathbf{N}} \longrightarrow C^{\mathbf{N}^+}$ which sends an object $X$ in $C^{\mathbf{N}}$ to $X^+$, defined by $X^+_{-1} = *$ and $X^+_n = X_n$ for $n \geq 0$.

Now we define the functor $e^*$ as the composite:

$$C^{\mathbf{N}} \times Cof(\mathbf{N}^{\mathbf{N}}) \xrightarrow{+\times -} C^{\mathbf{N}^+} \times Cof((\mathbf{N}^+)^{\mathbf{N}}) \xrightarrow{e} C^{\mathbf{N}},$$

so $e^*(X,\varphi) = e(X^+,\bar\varphi) = X^+{}_*\bar\varphi$; we also use the shorter notation $e^*(X,\varphi) = X^*\varphi$. For a fixed $\varphi$, we have a functor $-^*\varphi \colon C^{\mathbf{N}} \longrightarrow C^{\mathbf{N}}$ and for a fixed $X$, we have the contravariant functor $X^*- \colon Cof(\mathbf{N}^{\mathbf{N}}) \longrightarrow C^{\mathbf{N}}$.

2.5. Proposition. *Let $C$ be a category with a final object.*

*1) If $\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})$, then $-_*\varphi$ is left adjoint to $-^*\varphi$; that is, there is a natural transformation $C^{\mathbf{N}}(X_*\varphi, Y) \cong C^{\mathbf{N}}(X, Y^*\varphi)$ that will be denoted by $f \to f^b, g^\# \leftarrow g$.*

*2) If $\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})$, the following diagram is commutative*

$$\begin{tikzcd}
X \arrow[r, "f^b"] & Y^*\varphi \\
X_*\varphi \arrow[u, "X_*{}^{\varphi}_{id}"] \arrow[r, "f"'] & Y \arrow[u, "Y^*{}^{\varphi}_{id}"']
\end{tikzcd}$$

*3) If $\varphi, \psi \in In(\mathbf{N}^{\mathbf{N}}_{id})$, $\varphi \leq \psi$, for the diagram $X_*\psi \xrightarrow{X_*{}^{\psi}_{\varphi}} X_*\varphi \xrightarrow{f} Y$ we have that $(f(X_*{}^{\psi}_{\varphi}))^b = (Y^*{}^{\psi}_{\varphi})f^b$.*

Proof. 1) : We note that if $\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})$, $\bar\varphi\varphi = in \colon \mathbf{N} \to \mathbf{N}^+$. Therefore for a given object $Y$ in $C^{\mathbf{N}}$, we have

$$(Y^*\varphi)_*\varphi = (Y^+{}_*\bar\varphi)_*\varphi = Y^+{}_*(\bar\varphi\varphi) = Y^+{}_*in = Y$$

Define $^\# \colon C^{\mathbf{N}}(X, Y^*\varphi) \longrightarrow C^{\mathbf{N}}(X_*\varphi, Y)$ by $g^\# = g_*\varphi$. It is clear that the counit transformation is, in this case, the identity.

To define the inverse transformation $^b \colon C^{\mathbf{N}}(X_*\varphi, Y) \longrightarrow C^{\mathbf{N}}(X, Y^*\varphi)$, we have that $\varphi^+\bar\varphi \leq in$, since $\varphi$ is injective and $\varphi \geq id$. We also have that

$$X = X^+{}_*in$$
$$(X_*\varphi)^*\varphi = (X_*\varphi)^+{}_*\bar\varphi = (X^+{}_*\varphi^+)_*\bar\varphi = X^+{}_*\varphi^+\bar\varphi$$

Now, given $f \colon X_*\varphi \to Y$, define $f^b = (f^*\varphi)(X_*{}^{in}_{\varphi^+\bar\varphi})$; that is, it is the composite

$$X \xrightarrow{X^+{}_*{}^{in}_{\varphi^+\bar\varphi}} (X_*\varphi)^*\varphi \xrightarrow{f^*\varphi} Y^*\varphi\,.$$

By considering the formulas

$(f^b)^{\#} = [(f^*\varphi)(X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi})]^{\#} = ((f^*\varphi)_*\varphi)((X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi})_*\varphi) = f(X^{+}{}_{*}{}^{in\ \varphi}_{\varphi^+ in}) = f,$

$(g^{\#})^b = (g_*\varphi)^b = ((g_*\varphi)^*\varphi)(X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi}) = ((Y^*\varphi)^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi})g =$

$= ((Y^{+}{}_{*}\bar\varphi)^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi})g = ((Y^+)^{+}{}_{*}{}^{\bar\varphi^+ in}_{\bar\varphi^+\varphi^+\bar\varphi})g = ((Y^+)^{+}{}_{*}{}^{\bar\varphi^+ in}_{in^+\bar\varphi})g = g,$

it follows that the transformations above define a natural isomorphism.

2) The commutativity of the diagram is proved by the following equalities

$f^b(X_{*}{}^{\varphi}_{id}) = (f^{+}{}_{*}\bar\varphi)(X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi})((X_{*}{}^{\varphi}_{id})^{+}{}_{*}\ in) = (f^{+}{}_{*}\bar\varphi)((X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi}))(X^{+}{}_{*}{}^{\varphi^+ in}_{in}) =$

$= (f^{+}{}_{*}\bar\varphi)(X^{+}{}_{*}{}^{\varphi^+ in}_{\varphi^+\bar\varphi}) = (f^{+}{}_{*}\bar\varphi)((X^{+}{}_{*}\varphi^+)_{*}{}^{in}_{\bar\varphi}) = (f^{+}{}_{*}\bar\varphi)((X_*\varphi)^{+}{}_{*}{}^{in}_{\bar\varphi}) = (Y^{+}{}_{*}{}^{in}_{\bar\varphi})f =$

$= (Y^{*}{}^{\varphi}_{id})f.$

3) It follows from the following relations

$(f(X_{*}{}^{\psi}_{\varphi}))^b = ((f(X_{*}{}^{\psi}_{\varphi}))^{+}{}_{*}\bar\psi)(X^{+}{}_{*}{}^{in}_{\psi^+\bar\psi}) = ((f^+(X^{+}{}_{*}{}^{\psi^+}_{\varphi^+}))_{*}\ \bar\psi)(X^{+}{}_{*}{}^{in}_{\psi^+\bar\psi})$

$= (f^{+}{}_{*}\bar\psi)(X^{+}{}_{*}{}^{\psi^+\bar\psi}_{\varphi^+\bar\psi})(X^{+}{}_{*}{}^{in}_{\psi^+\bar\psi}) = (f^{+}{}_{*}\ \bar\psi)(X^{+}{}_{*}{}^{in}_{\psi^+\bar\psi})$

$= (f^{+}{}_{*}\ \bar\psi)(X^{+}{}_{*}{}^{\varphi^+\bar\varphi}_{\varphi^+\bar\psi})(X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi}) = (f^{+}{}_{*}\ \bar\psi)((X_*\varphi)^{+}{}_{*}{}^{\bar\varphi}_{\bar\psi})(X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi})$

$= (Y^{+}{}_{*}{}^{\bar\varphi}_{\bar\psi})(f^{+}{}_{*}\bar\varphi)(X^{+}{}_{*}{}^{in}_{\varphi^+\bar\varphi}) = (Y^{*}{}^{\psi}_{\varphi})\ f^b$

∎

2.6. Lemma. *Let $C$ be a category with final object $*$, and $\varphi, \psi \in In(\mathbf{N}^{\mathbf{N}}_{id})$, then*

*1)* $(\_^*\varphi)^*\psi = \_^*(\psi\varphi)$

*2)* $(Y^*\varphi)^{*}{}^{\psi}_{id} = Y^{*}{}^{\varphi\psi}_{\varphi}$

Proof. It is easy to check that $\bar\varphi^+\bar\psi = in^+\overline{\psi\varphi}$. Therefore we have:

1) $(Y^*\varphi)^*\psi = (Y^{+}{}_{*}\bar\varphi)^*\psi = (Y^{+}{}_{*}\bar\varphi)^{+}{}_{*}\bar\psi = (Y^+)^{+}{}_{*}\ \bar\varphi^{+}{}_{*}\ \bar\psi =$

$(Y^+)^{+}{}_{*}\ \bar\varphi^+\bar\psi = (Y^+)^{+}{}_{*}\ in^+\overline{\psi\varphi} =$

$= Y^{+}{}_{*}\ \overline{\psi\varphi} = Y^*(\psi\varphi).$

2) $(Y^*\varphi)^{*}{}^{\psi}_{id} = (Y^{+}{}_{*}\bar\varphi)^{*}{}^{\psi}_{id} = (Y^{+}{}_{*}\bar\varphi)^{+}_{*}{}^{in}_{\bar\psi} = (Y^+)^{+}{}_{*}\ \bar\varphi^{+}{}_{*}{}^{in}_{\bar\varphi}$

$= (Y^+)^{+}{}_{*}{}^{\bar\varphi^+ in}_{\bar\varphi^+\bar\psi} = (Y^+)^{+}{}_{*}{}^{in^+\bar\varphi}_{in^+\overline{\psi\varphi}} = Y^{+}{}_{*}{}^{\bar\varphi}_{\overline{\psi\varphi}} = Y^{*}{}^{\psi\varphi}_{\varphi}.$ ∎

Now we define a class $\Gamma$ of morphisms in $C^{\mathbf{N}}$ that admits a calculus of left fractions. The category $\Gamma^{-1}C^{\mathbf{N}}$ will be equivalent to $towC = pro_{\mathbf{N}}C$.

Consider the class $\Gamma$ defined by the morphisms of the form $Y^{*}{}^{\varphi}_{id}\colon Y \longrightarrow Y^*\varphi$, where $Y$ is an object in $C^{\mathbf{N}}$ and $\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})$ ($C$ has a final object).

2.7. Proposition. $\Gamma$ *admits a calculus of left fractions.*

Proof. a) It is clear that $Y^{*}{}^{id}_{id} = id_Y$.

b) By Lemma 2.6, we have

$$((Y^*\varphi)^{*}{}^{\psi}_{id})(Y^{*}{}^{\varphi}_{id}) = (Y^{*}{}^{\psi\varphi}_{\varphi})(Y^{*}{}^{\varphi}_{id}) = Y^{*}{}^{\psi\varphi}_{id}$$

c) For each diagram $X^*\varphi \xleftarrow{X^*{}^{\varphi}_{id}} X \xrightarrow{f} Y$, we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "X^*{}^{\varphi}_{id}"'] & Y \arrow[d, "Y^*{}^{\varphi}_{id}"] \\ X^*\varphi \arrow[r, "f^*\varphi"'] & Y^*\varphi \end{tikzcd}$$

where $Y^*{}^{\varphi}_{id}$ is in $\Gamma$.

d) Consider a diagram $X \xrightarrow{X^*{}^{\varphi}_{id}} X^*\varphi \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that $f(X^*{}^{\varphi}_{id}) = g(X^*{}^{\varphi}_{id})$.

Applying Lemma 2.6 we have
$(Y^*{}^{\varphi}_{id})f = (f^*\varphi)((X^*\varphi)^*{}^{\varphi}_{id}) = (f^*\varphi)(X^*{}^{\varphi\varphi}_{\varphi}) = (f^*\varphi)((X^*{}^{\varphi}_{id})^*\varphi) = (f(X^*{}^{\varphi}_{id}))^*\varphi =$
$= (g(X^*{}^{\varphi}_{id}))^*\varphi = (g^*\varphi)(X^*{}^{\varphi\varphi}_{\varphi}) = (g^*\varphi)(X^*\varphi)^{\varphi}_{id} = (Y^*{}^{\varphi}_{id})g.$
Therefore there exists $Y^*{}^{\varphi}_{id}$ in $\Gamma$ satisfying the desired relation. ∎

If we consider the diagram

$$\begin{tikzcd} C^{scd\mathbf{N}} \arrow[d] & C^{\mathbf{N}} \arrow[l] \arrow[d] \\ pro_{\mathbf{N}}C & \Gamma^{-1}\, C^{\mathbf{N}} \end{tikzcd}$$

we can see that a morphism $X^*{}^{\varphi}_{id}\colon X \longrightarrow X^*\varphi$ of $\Gamma$ has an inverse $\tau^{\varphi}_{id}(X)$ in $pro_{\mathbf{N}}C$. Define $\tau^{\varphi}_{id}\,(X) = (\varphi, id_X)$ in $C^{scd\mathbf{N}}$.

2.8. Lemma. *The morphisms $X^*{}^{\varphi}_{id}$ and $\tau^{\varphi}_{id}(X)$ give an isomorphism in the category $pro_{\mathbf{N}}C = towC$.*

Proof. We have that

$$(id_{\mathbf{N}}, X^*{}^{\varphi}_{id})(\varphi, id_X) = (\varphi, X^*{}^{\varphi}_{id})$$

Since the following diagram is commutative

$$\begin{tikzcd}
(X^*\varphi)_*\varphi \arrow[r, "(X^*\varphi)_*{}^{\varphi}_{\varphi}"] \arrow[d, "(X^*\varphi)_*{}^{\varphi}_{id}"'] & (X^*\varphi)_*\varphi \arrow[d, "(X^*\varphi)_*{}^{\varphi}_{id}"] \\
X^*\varphi \arrow[r, "id"] & X^*\varphi
\end{tikzcd}$$

we have that $(\varphi, X^*{}^{\varphi}_{id}) = (id_{\mathbf{N}}, id_{X^*\varphi})$.

On the other hand we have

$(X^*{}^{\varphi}_{id})_*\varphi = (X^+{}_{*\bar{\varphi}}^{in})_*\varphi = X^+{}_{*\bar{\varphi}\varphi}^{in\varphi} = X^+{}_{*in}^{in}{}^{\varphi}_{id} = X^*{}^{\varphi}_{id}$ ,

$(\varphi, id_X)(id_{\mathbf{N}}, X^*{}^{\varphi}_{id}) = (\varphi, (X^*{}^{\varphi}_{id})_*\varphi) = (\varphi, X^*{}^{\varphi}_{id})$.

We have already seen in the proof of Lemma 2.3 that $(\varphi, X_*{}^{\varphi}_{id}) = (id_{\mathbf{N}}, id_X)$. ∎

2.9. Theorem. *For a category $C$ with final object, the induced functor $\Gamma^{-1}C^{\mathbf{N}} \longrightarrow pro_{\mathbf{N}}C$ is an equivalence of categories.*

Proof. It suffices to dualize the proof of Theorem 2.4. ∎

2.10. Remark. 1) We can also prove this theorem taking into account the definition of the hom–set, see [G–Z], and Proposition 2.5

$$\Gamma^{-1}C^{\mathbf{N}}(X,Y) = \operatorname*{colim}_{\varphi\in In(\mathbf{N}^{\mathbf{N}}_{id})} C^{\mathbf{N}}(X, Y^*\varphi) \cong \operatorname*{colim}_{\varphi\in In(\mathbf{N}^{\mathbf{N}}_{id})} C^{\mathbf{N}}(X_*\varphi, Y) \cong pro_{\mathbf{N}}C(X,Y),$$

2) The equivalence of categories $C^{\mathbf{N}}\Sigma^{-1} \longrightarrow \Gamma^{-1}C^{\mathbf{N}}$ is given by

$$(X \xleftarrow{X_*{}^{\varphi}_{id}} X_*\varphi \xrightarrow{f} Y) \longrightarrow (X \xrightarrow{f^b} Y^*\varphi \xrightarrow{Y^*{}^{\varphi}_{id}} Y).$$

Consider the class $\Sigma_C$ of morphisms of $C^{\mathbf{N}}$ of the from $X_*{}^{\varphi}_{id}\colon X_*\varphi \longrightarrow X$. If $D$ is a finite category and $C^D$ denotes the category of functors of the form $D \to C$ we can consider the class $\Sigma_{C^D}$ of morphisms of $(C^D)^{\mathbf{N}}$ of the form $A_*\varphi \to A$. The corresponding category of fractions is denoted by $tow(C^D)$. Notice that we also have the equivalence of categories $(C^D)^{\mathbf{N}} \to (C^{\mathbf{N}})^D$ and the functor $C^{\mathbf{N}} \to towC$ induces the natural functor $(C^{\mathbf{N}})^D \to (towC)^D$.

With this notation we have the following result:

2.11. PROPOSITION. *If $D$ is a finite category, then there is a diagram*

$$\begin{tikzcd} {(C^D)}^{\mathbf{N}} \arrow[r] \arrow[d] & {(C^{\mathbf{N}})}^D \arrow[d] \\ tow(C^D) \arrow[r] & {(towC)}^D \end{tikzcd}$$

*which is commutative up to isomorphism and is such that the induced functor $tow(C^D) \longrightarrow (towC)^D$ is a full embedding.*

PROOF. Let $X$ denote an object of ${(C^D)}^{\mathbf{N}}$ and the corresponding object in ${(C^{\mathbf{N}})}^D$. If $X$ is an object in ${(C^D)}^{\mathbf{N}}$, $X(n)$ denotes a diagram of $C^D$ and if $X$ is thought of as an object in ${(C^{\mathbf{N}})}^D$, then $X_d$ is an object in $C^{\mathbf{N}}$. We note that $X(n)_d = X_d(n)$.

Now suppose that $X, Y$ are objects in ${(C^D)}^{\mathbf{N}}$ (or in ${(C^{\mathbf{N}})}^D$) and $f\colon X \to Y$ is a morphism in $(towC)^D$. Then for each $d$ an object of $D$, we can represent $f_d$ for all $d$ by $X_d \longleftarrow (X_d)_*\varphi^d \xrightarrow{f'_d} Y_d$. By considering a map $\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})$ such that $\varphi \geq \varphi^d,\ d \in Ob\ D$, we can represent $f_d$ for all $d$ by $X_d \longleftarrow (X_d)_*\varphi \xrightarrow{f'_d} Y_d$. Then we have that $X \longleftarrow X_*\varphi$ is a morphism in ${(C^D)}^{\mathbf{N}}$. However, if $d_0 \xrightarrow{\alpha_0} d_1$ is a morphism in $D$, then

$$\begin{tikzcd} X_{d_0 *}\varphi \arrow[r, "f'_{d_0}"] \arrow[d, "(X_{\alpha_0})_*\varphi"'] & Y_{d_0} \arrow[d, "Y\alpha_0"] \\ X_{d_1 *}\varphi \arrow[r, "f'_{d_1}"'] & Y_{d_1} \end{tikzcd}$$

is only commutative in $towC$. Nevertheless we can choose $\psi_{\alpha_0} \in In(\mathbf{N}^{\mathbf{N}}_{id})$ such that

$$\begin{tikzcd} X_{d_0 *}\varphi_*\psi_{\alpha_0} \arrow[r, "f''_{d_0}"] \arrow[d] & Y_{d_0} \arrow[d] \\ X_{d_1 *}\varphi_*\psi_{\alpha_0} \arrow[r, "f''_{d_1}"'] & Y_{d_1} \end{tikzcd}$$

is commutative in $C^{\mathbf{N}}$. Since the set of morphisms of $D$ is finite, we finally obtain representatives maps $X_d \longleftarrow X_{*}\varphi_{*}\psi_{\alpha_0 *}\cdots_{*}\psi_{\alpha_r} \xrightarrow{\bar{f}_d} Y_d$ such that $X \longleftarrow X_{*}\varphi_{*}\psi_{\alpha_0 *}\cdots{}_{*}\psi_{\alpha_r} \xrightarrow{f} Y$ is a diagram in $(C^D)^{\mathbf{N}}$.

This diagram represents a morphisms from $X$ to $Y$ in $tow(C^D)$ that is sent to $f\colon X \to Y$ by the functor $tow(C^D) \longrightarrow (tow\ C)^D$.

Now if $f, g\colon X \to Y$ are maps in $(C^D)^{\mathbf{N}}$ such that $f = g$ in $(tow\ C)^D$, we have for each $d \in Ob\ D$ a map $\varphi^d \in In(\mathbf{N}^{\mathbf{N}}_{id})$ such that $f_d(X_{d*}{}^{\varphi^d}_{id}) = g_d(X_{d*}{}^{\varphi^d}_{id})$. If $\varphi \geq \varphi^d$, $d \in Ob\ D$, then $f_d(X_{d*}{}^{\varphi}_{id}) = g_d(X_{d*}{}^{\varphi}_{id})$. Therefore $f(X_{*}{}^{\varphi}_{id}) = g(X_{*}{}^{\varphi}_{id})$. This implies that $f = g$ in $tow(C^D)$. ∎

2.12. Remark. Meyer [Mey] has proved that if the category $C$ has finite limits, then the functor $pro(C^D) \longrightarrow (proC)^D$ is an equivalence of categories.

## 3. Preliminaries on monoids, near-rings and rings

In this section, we establish the notation and properties of monoids, near-rings and rings that will be used in next sections. We usually consider the categories of sets, pointed sets, groups and abelian groups which are denoted by *Set*, $Set_*$, *Grp* and *Ab*, respectively.

A monoid consists of a set $M$ and an associative multiplication $\cdot : M \times M \longrightarrow M$ with unit element 1 ($1{\cdot}m = m = m{\cdot}1$, for every $m \in M$). If $M$ has also a zero element 0 ($m{\cdot}0 = 0 = 0{\cdot}m$, for every $m \in M$) it will be called a 0-monoid.

A set $R$ with two binary operations '+' and '·', is a unitary (left) near–ring if $(R, +)$ is a group (the additive notation does not imply commutativity), $(R, \cdot)$ is a semigroup and the operations satisfy the left distributive law:

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad x, y, z \in R.$$

A near–ring $R$ satisfies that $x \cdot 0 = 0$ and $x \cdot (-y) = -(x \cdot y)$, but in general, it is not true that $0 \cdot x = 0$ for all $x \in R$. If the near-ring also satisfies the last condition it is called a zero-symmetric near-ring. In this paper we will only work with zero-symmetric unitary near-rings. In this case, $(R, \cdot)$ is a 0-monoid.

If a zero-symmetric unitary near-ring also satisfies the right distributive law:

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad x, y, z \in R$$

then $(R, +)$ is abelian and $R$ becomes a unitary ring.

3.1. Example. If $\mathcal{C}$ is a category and $X$ is an object of $\mathcal{C}$, then $\mathcal{C}(X, X)$ is a monoid with the composition of morphisms: $(g, f) \to g \cdot f$. In next sections $\mathcal{C}$ will be one of the categories *proC* or *towC*.

3.2. Example. If $\mathcal{C}$ is a category with a zero object, the monoid $\mathcal{C}(X, X)$ has a zero element $0\colon X \to X$ and $\mathcal{C}(X, X)$ is a 0–monoid. If $C$ has a zero object, the categories *proC* and *towC* also have zero objects.

3.3. Example. Let $F$ be a free group generated by a set $X$, then the set of endomorphisms of $F$, $End(F)$, becomes a zero-symmetric unitary left near-ring if the operation $+$ is defined by

$$(f+g)x = fx + gx, \quad f, g \in End(F), \quad x \in X.$$

3.4. Example. If $A$ is an abelian group or an object in an abelian category, then $End(F)$ is an unitary ring.

Let $M$ be a monoid and $\mathcal{C}$ a category. A left $M$–object $X$ in $\mathcal{C}$ consists of an object $X$ of $\mathcal{C}$ and a monoid homomorphism $M \longrightarrow \mathcal{C}(X,X) : m \to \tilde{m} : X \longrightarrow X$. If $M$ is a 0–monoid and $\mathcal{C}$ has a zero object, we will also assume that an $M$–object $X$ in $\mathcal{C}$ satisfies the additional condition $\tilde{0} = 0$. We denote by ${}_M\mathcal{C}$ the category whose objects are the (left) $M$–objects in $\mathcal{C}$. By considering monoid "antimorphisms" $M \longrightarrow \mathcal{C}(X,X)$ we have the notion of right $M$–object in $\mathcal{C}$ and the category $\mathcal{C}_M$.

For the case $\mathcal{C} = Set$ ($\mathcal{C} = Set_*$) we have the notion of $M$-set ($M$-pointed set) and the categories ${}_MSet$, $Set_M$ (${}_MSet_*$, $Set_{*M}$). If $X$ is a group, then $Set_*(X,X)$ has a natural structure of zero-symmetric unitary right near-ring. If $R$ is a zero-symmetric unitary right near-ring, a structure of left $R$-group on $X$ (left near-module) is given by a near-ring homomorphism $R \longrightarrow Set_*(X,X)$. If $R$ is a zero-symmetric unitary left near-ring and $R \longrightarrow Set_*(X,X)$ is a group homomorphism and a monoid antimorphism, then X is said to have a structure of right $R$-group. We denote by ${}_RGrp$ the category of left $R$-groups and by $Grp_R$ the category of right $R$-groups.

If $X$ is an abelian group, then $Ab(X,X)$ has a natural structure of unitary ring. If $R$ is a ring, a structure of left $R$-abelian group ($R$-module) is given by a ring homomorphism $R \longrightarrow Ab(X,X)$. If $R \longrightarrow Ab(X,X)$ is a group homomorphism and a monoid antimorphism, then $X$ is said to be a right $R$-abelian group ($R$-module). We denote by ${}_RAb$ the category of left $R$-abelian groups and by $Ab_R$ the category of right $R$-abelian groups.

3.5. Example. Let $\mathcal{C}$ be a category. For each object $X$ of $\mathcal{C}$, we have the monoid $End(X) = \mathcal{C}(X,X)$ and $\mathcal{C}(X,-) : \mathcal{C} \longrightarrow Set_{End(X)}$ is a functor which associates to an object $Y$ the right $End(X)$–object defined by $End(X) \longrightarrow Set(\mathcal{C}(X,Y), \mathcal{C}(X,Y))$ : $\varphi \longrightarrow \tilde{\varphi}$, $\tilde{\varphi}(f) = f\varphi$, $f \in \mathcal{C}(X,Y)$. If $\mathcal{C}$ has a zero object, we also have the functor: $\mathcal{C}(X,-) \colon \mathcal{C} \longrightarrow Set_{*End(X)}$.

3.6. Example. Let $F$ be a free group generated by the set $X$. We have noted in Example 3) above that $End(F)$ is a left near-ring. Is is easy to see that for any group $Y$, $Grp(F,Y)$ has a natural structure of right $End(F)$-group. Therefore there is an induced functor $Grp(F,-) : Grp \longrightarrow Grp_{End(F)}$.

3.7. Example. If $X$ is an object in an abelian category $\mathcal{A}$, then $\mathcal{A}(X,-)$ defines a functor from $\mathcal{A}$ to the category of $End(X)$-abelian groups ( $End(X)$-modules).

Recall that in this paper we are using the following unified notation. We denote by $C$ one of the categories: $Set$, $Set_*$, $Grp$, $Ab$. The small projective generator of $C$ is denoted by $G$. We also denote by $*$, $S^0$, $\mathbf{Z}$, $\mathbf{Z}_a$ the corresponding generators of these categories.

Because the categories of the examples above have some common properties, we will use the following notation:

1) If $C = Set$ and $R$ is a monoid, $C_R$ denotes the category of right $R$-sets.

2) If $C = Set_*$ and $R$ is a 0-moniod, $C_R$ denotes the category of right $R$-pointed sets.

3) If $C = Grp$ and $R$ is a zero-symmetric unitary left near-ring, $C_R$ denotes the category of right R-groups (R-near-modules).

4) If $C = Ab$ and $R$ is an unitary ring, $C_R$ denotes the category of right $R$-abelian groups ($R$-modules).

It is interesting to note that $C_R$ and $C$ are algebraic categories and there is a natural forgetful functor $U: C_R \longrightarrow C$ which has a left adjoint functor $F = - \otimes R: C \longrightarrow C_R$. If $C = Set$ and $X$ is a set, then $X \otimes R = X \times R$ and if $C = Set_*$, then $X \otimes R = X \times R/((* \times R) \cup (X \times 0))$. It is also easy to define $- \otimes R$ for the cases $C = Grp$ and $C = Ab$.

If the functor $F: \mathcal{A} \to \mathcal{B}$ is left adjoint to the functor $U: \mathcal{B} \to \mathcal{A}$, then it is well known that $F$ preserves colimits and that $U$ preserves limits. A functor $U: \mathcal{B} \to \mathcal{A}$ reflects finite limits if it verifies the following property: If $X$ is the "cone" over a finite diagram $D$ in $\mathcal{B}$ and $UX$ is the limit of $UD$, then $X$ is the limit of $D$.

We summarise some properties of the functors above in the following:

3.8. Proposition. *The forgetful functor $U: C_R \longrightarrow C$ has a left adjoint functor $F = - \otimes R: C \longrightarrow C_R$. Moreover, the functor $U$ preserves and reflects finite limits, in particular if $Uf$ is an isomorphism, then $f$ is also isomorphism.*

Proof. It suffices to check that $U$ reflects finite products and difference kernels. If $Y_1, Y_2$ are objects in $C_R$, then $UY_1 \times UY_2$ admits an action of $R$ defined by $(y_1, y_2)r = (y_1 r, y_2)r$, if $r \in R$. Now it is easy to check that for the different cases, $C = Set, Set_*, Grp, Ab$, this action satisfies the necessary properties to define an object $Y_1 \times Y_2$ in $C_R$ such that $U(Y_1 \times Y_2) = UY_1 \times UY_2$. Similarly if $f, g: Y \to Y'$ are morphisms in $C_R$, then the difference kernel $K(Uf, Ug)$ is defined by $K(Uf, Ug) = \{x \in UY \mid Ufx = Ugx\}$. In this case the action of R on Y induces an action on $K(Uf, Ug)$ that satisfies the necessary properties, and therefore defines an object $K(f, g)$ such that $UK(f, g) \cong K(Uf, Ug)$. ∎

Given a morphism $R_0 \to R_1$, there is an induced functor $V: C_{R_1} \longrightarrow C_{R_0}$ which has a left adjoint functor $-\otimes_{R_0} R_1 : C_{R_0} \longrightarrow C_{R_1}$. It is not hard to give a more explicit definition of the functor $-\otimes_{R_0} R_1$ for the cases $C = Set,\ \ Set_*,\ \ Grp,\ \ Ab$.

In next sections, we will consider the properties of the following construction to study the inverse limit functor.

Let $s$ be an element of $R$ (if $C = Gps$ and $R$ a left near-module, we also assume that $s$ is a right distributive element: $(x+y)s = xs+ys$), and let X be an object in $C_R$, define

$$F_s X = \{x \in X \mid x \cdot s = x\}.$$

This gives a functor $F_s: C_R \longrightarrow C$ which has a left adjoint $-\otimes^s R: C \longrightarrow C$ defined as follows: Let $X$ be an object of C, the functor $-\otimes R: C \longrightarrow C_R$ carries $X$ to $X \otimes R$. Consider on $X \otimes R$ the equivalence relation compatible with the corresponding algebraic structure and generated by the relations $x \otimes r \sim x \otimes sr$. Denote the quotient object by $X \otimes^s R$, and the equivalence class of $x \otimes r$ by $x \otimes^s r$. We summarise this construction in the following

3.9. PROPOSITION. *The functor $- \otimes^s R: C \longrightarrow C_R$ is left adjoint to $F_s: C_R \longrightarrow C$.*

## 4. Brown's construction.

In this section we define the $\mathcal{P}$ functor for the categories of pro-sets, pro-pointed sets, pro-groups and pro-abelian groups. As in the section above, $C$ denotes one of the following categories: $Set$, $Set_*$, $Grp$, $Ab$.

Because the category $C$ has products and sums, then we can define the functors $c: C \longrightarrow C^{\mathbf{N}}$ and $p: C^{\mathbf{N}} \longrightarrow C$ by the formulas

$$(cX)_i = \sum_{j \geq i} X_j, \quad X_j = X, \quad j \geq i,$$

$$pY = \prod_{i=0}^{+\infty} Y_i$$

It is easy to check that $C^{\mathbf{N}}(cX, Y) \cong C(X, pY)$, therefore we have:

4.1. PROPOSITION. *The functor $c: C \longrightarrow C^{\mathbf{N}}$ is left adjoint to $p: C^{\mathbf{N}} \longrightarrow C$.*

Associated with the generator $G$ of $C$, we have the pro-object $cG$ and the endomorphism set $\mathcal{P}cG = proC(cG, cG)$ which has the following properties:

1) If $C = Set$, the morphism composition gives to $\mathcal{P}c*$ a monoid structure.
2) If $C = Set_*$, $\mathcal{P}cS^0$ is a 0-monoid (see section 3 .)
3) If $C = Grp$, $\mathcal{P}c\mathbf{Z}$ is a zero-symmetric unitary left near-ring.
4) If $C = Ab$, $\mathcal{P}c\mathbf{Z}_a$ is a ring.

For any object $X$ of $proC$, we consider the natural action

$$proC(cG, X) \times proC(cG, cG) \longrightarrow proC(cG, X)$$

which applies $(f, \varphi)$ to $f\varphi$, if $f \in proC(cG, X)$ and $\varphi \in proC(cG, cG)$.

The morphism set $proC(cG, X)$ has the following properties:

1) If $C = Set$, $proC(cG, X)$ admits a natural structure of $\mathcal{P}cG$-set. Thus the action satisfies

$$(f\alpha)\beta = f(\alpha\beta)$$
$$f1 = f$$

$f \in proC(cG, X)$, $\alpha$, $\beta$, $1 \in proC(cG, cG)$.

2) If $C = Set_*$, $proC(cG, X)$ and $proC(cG, cG)$ have zero morphisms that satisfy

$$f0 = 0, \quad f \in proC(cG, X),$$
$$0\alpha = 0, \quad \alpha \in \mathcal{P}cG;$$

that is, $\mathcal{P}cG$ is a 0-monoid (see section 3) and $proC(cG, X)$ is a $\mathcal{P}cG$-pointed set.

3) If $C = Grp$, we also have that $proC(cG, X)$ has a group structure and the action satisfies the left distributive law:

$$f(\alpha + \beta) = f\alpha + f\beta, \quad f \in proC(cG, X), \quad \alpha, \beta \in \mathcal{P}cG$$

Notice that the sum $+$ need not be commutative. In this case, $\mathcal{P}cG$ becomes a zero-symmetric unitary left near-ring and $proC(cG, X)$ is a right $\mathcal{P}cG$-group (near-module), see [Mel] and [Pilz].

4) If $C = Ab$, we also have a right distributive law:

$$(f + g)\alpha = f\alpha + g\alpha, \quad f, g \in proC(cG, X), \quad \alpha \in \mathcal{P}cG.$$

Now $\mathcal{P}cG$ becomes a unitary ring and $proC(cG, X)$ is a right $\mathcal{P}cG$-abelian group ($\mathcal{P}cG$-module).

In order to have a unified notation, $C_{\mathcal{P}cG}$ denotes one of the following categories:

1) If $C = Set$, $C_{\mathcal{P}cG}$ is the category of $\mathcal{P}cG$-sets.

2) If $C = Set_*$, $C_{\mathcal{P}cG}$ is the category of $\mathcal{P}cG$-pointed sets.

3) If $C = Grp$, $C_{\mathcal{P}cG}$ is the category of $\mathcal{P}cG$-groups (near-modules).

4) If $C = Ab$, $C_{\mathcal{P}cG}$ is the category of $\mathcal{P}cG$-abelian groups (modules).

Using the notation above we can define a functor $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ as the representable functor

$$\mathcal{P}X = proC(cG, X)$$

together with the natural action of $\mathcal{P}cG$, where $G$ is the small projective generator of $C$. For the full subcategory $towC$ we will also consider the restriction functor $\mathcal{P}\colon towC \longrightarrow C_{\mathcal{P}cG}$.

Because $C$ has sums, products and a final object $*$, for any object $Y$ of $C$, we can consider the direct system

$$\prod_{i\geq 0} Y \rightarrow (\prod_{i\geq 1} Y) \times * \rightarrow (\prod_{i\geq 2} Y) \times * \times * \rightarrow \cdots$$

where the bonding maps are induced by the identity $id\colon Y \rightarrow Y$ and the zero map $Y \rightarrow *$. The reduced product $IY$ of $Y$ is defined to be the colimit of the direct system above. We also recall the forgetful functor $U\colon C_{\mathcal{P}cG} \longrightarrow C$ considered in section 3 which will be used in the following

4.2. Proposition. *The functor $U\mathcal{P}\colon proC \longrightarrow C$ has the following properties:*

*1) If $X = \{X_j\}$ is an object of $proC$, then*

$$U\mathcal{P}X \cong \lim_j IX_j.$$

*2) If $X = \{X_j \mid j \in J\}$ is an object of $proC$, where $J$ is a strongly cofinite directed set, then*

$$U\mathcal{P}X \cong \operatorname*{colim}_{\varphi\in\mathbf{N}^J} C^J(cG_*\varphi, X).$$

*3) If $X$ is an object in $towC$, then*

$$U\mathcal{P}X \cong \operatorname*{colim}_{\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})} p(X^*\varphi).$$

PROOF. For 1), it suffices to consider the definition of the hom–set in $proC$:

$$\begin{aligned} U\mathcal{P}X &= \lim_j \ \operatorname*{colim}_i C((cG)_i, X_j) \\ &\cong \lim_j \ \operatorname*{colim}_i \prod_{k \geq i} X_j \\ &\cong \lim_j IX_j \end{aligned}$$

2) follows since $\mathbf{N}$, $J$ are strongly cofinite directed sets.

3) By Remark 1) after Theorem 2.9 and Proposition 4.1, we get

$$\begin{aligned} U\mathcal{P}X &= towC(cG, X) \\ &\cong \operatorname*{colim}_{\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})} C^{\mathbf{N}}(cG, X^*\varphi) \\ &\cong \operatorname*{colim}_{\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})} C(G, p(X^*\varphi)) \\ &\cong \operatorname*{colim}_{\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})} p(X^*\varphi) \end{aligned}$$

■

4.3. PROPOSITION. *The functor $\mathcal{P}: proC \longrightarrow C_{\mathcal{P}cG}$ preserves finite limits.*

PROOF. We have that $U\mathcal{P}$ preserves finite limits since $U\mathcal{P} = proC(cG, -)$ is a representable functor, see [Pa; Th 1, Sect 9, Ch.2]. By Proposition 3.8, we have that $U$ reflects finite limits. Therefore we get that $\mathcal{P}$ preserves finite limits. ■

Now we recall that Grossman in [Gr.3] proved that the functor $U\mathcal{P}: towC \longrightarrow C$ reflects isomorphisms. Since $\mathcal{P}$ preserves finite limits we have:

4.4. THEOREM. *$\mathcal{P}: towC \longrightarrow C_{\mathcal{P}cG}$ is a faithful functor.*

PROOF. Let $f, g: X \to Y$ be two morphisms in $towC$. If we consider the difference kernel $i: K(f, g) \longrightarrow X$, the Proposition above implies that $\mathcal{P}K(f, g) \cong K(\mathcal{P}f, \mathcal{P}g)$. Suppose that $\mathcal{P}f = \mathcal{P}g$, then $K(\mathcal{P}f, \mathcal{P}g) \cong \mathcal{P}X$ and $\mathcal{P}i: \mathcal{P}K(f, g) \longrightarrow \mathcal{P}X$ is an isomorphism. Applying the forgetful functor $U: C_{\mathcal{P}cG} \longrightarrow C$, we have that $U\mathcal{P}i$ is an isomorphism. Now Grossman's result implies that $i$ is also an isomorphism. Therefore $f = g$. ■

4.5. REMARK. 1) Since $\mathcal{P}: towC \longrightarrow C_{\mathcal{P}cG}$ is a representable faithful functor, we have that $\mathcal{P}$ preserves monomorphisms and reflects monomorphisms and epimorphisms.

2) Notice that the proof given does not work for the larger category $proC$.

4.6. Proposition. *$\mathcal{P}\colon towC \longrightarrow C_{\mathcal{P}cG}$ and $U\mathcal{P}\colon towC \longrightarrow C$ preserve epimorphisms.*

Proof. Let $q'\colon X' \longrightarrow Y'$ be an epimorphism in $towC$, by the Remarks at the end of ChII, §2.3 of [M–S] it follows that $q'$ is isomorphic in $Maps(towC)$ to $q\colon X \longrightarrow Y$ where $q$ is a level map $\{q_i\colon X_i \longrightarrow Y_i\}$ and each $q_i\colon X_i \longrightarrow Y_i$ is a surjective map. Now we have that $U\mathcal{P}q = \mathrm{colim}_\varphi\, p(q_*\varphi)$, and since $-^*\varphi,\ \ p(-),\ \mathrm{colim}_\varphi$ preserve epimorphisms, we get that $U\mathcal{P}q$ and $\mathcal{P}q$ are epimorphisms. ∎

4.7. Definition. *Let $\mathcal{S}$ be a full subcategory of $proC$. An object $X$ in $\mathcal{S}$ is said to be admissible in $\mathcal{S}$ if for every $Y$ of $\mathcal{S}$ the transformation*

$$\mathcal{P}\colon proC(X,Y) \longrightarrow C_{\mathcal{P}cG}(\mathcal{P}X,\mathcal{P}Y)\colon f \to \mathcal{P}f,$$

*is bijective. If $\mathcal{S} = proC$, $X$ is said to be admissible.*

4.8. Proposition. *The object $cG$ is admissible.*

Proof. We have the following natural isomorphisms

$$\begin{aligned} proC(cG,X) &\cong U\mathcal{P}X \\ &\cong C(G,U\mathcal{P}X) \\ &\cong C_{\mathcal{P}cG}(G\otimes\mathcal{P}cG,\mathcal{P}X) \\ &\cong C_{\mathcal{P}cG}(\mathcal{P}cG,\mathcal{P}X) \end{aligned}$$

which send a map $f\colon cG \to X$ to $\mathcal{P}f\colon \mathcal{P}cG \to \mathcal{P}X$. ∎

4.9. Proposition. *Let $p\colon X \to Y$ be an epimorphism in $towC$. If $X$ is admissible in $towC$, then $Y$ is also admissible in $towC$.*

Proof. We can suppose that $p$ is a level map $\{p_i\colon X_i \longrightarrow Y_i\}$ such that each $p_i\colon X_i \longrightarrow Y_i$ is a surjective map. Let $X_i \underset{Y_i}{\times} X_i$ denote the equivalence relation associated with $p_i$; that is, $X_i \underset{Y_i}{\times} X_i = \{(x,x') \in X_i \times X_i \mid p_i x = p_i x'\}$. Then the diagram

$$X \underset{Y}{\times} X \;\underset{pr_2}{\overset{pr_1}{\rightrightarrows}}\; X \xrightarrow{\ p\ } Y$$

is a difference cokernel in $towC$, where $X \underset{Y}{\times} X = \{X_i \underset{Y_i}{\times} X_i\}$.

Let $Z$ be an object in $towC$. Given a morphism $\beta\colon \mathcal{P}Y \longrightarrow \mathcal{P}Z$ in $C_{\mathcal{P}cG}$, since $X$ is admissible in $towC$, there is a morphism $f\colon X \to Z$ in $towC$ such that $\beta\ \mathcal{P}p = \mathcal{P}f$.

$$\begin{gathered} \mathcal{P}(f\ pr_1) = \mathcal{P}f\ \mathcal{P}pr_1 = \beta\ \mathcal{P}p\ \mathcal{P}pr_1 = \beta\ \mathcal{P}(p\ pr_1) = \\ \beta\ \mathcal{P}(p\ pr_2) = \beta\ \mathcal{P}p\ \mathcal{P}pr_2 = \mathcal{P}f\ \mathcal{P}pr_2 = \mathcal{P}(f\ pr_2). \end{gathered}$$

Because $\mathcal{P}$ is faithful, it follows that $f\ pr_1 = f\ pr_2$. Now we can use that $p$ is a difference cokernel to obtain a morphism $g\colon Y \to Z$ such that $gp = f$. We have that $\mathcal{P}g\ \mathcal{P}p = \mathcal{P}f = \beta\ \mathcal{P}p$. By Proposition 4.6, $\mathcal{P}p$ is an epimorphism, then we have that $\mathcal{P}g = \beta$. This implies that $Y$ is also admissible in $towC$. ∎

4.10. Definition. *An object $X$ of $towC$ is said to be finitely generated if there is an (effective) epimorphism of the form* $\sum_{finite} cG \longrightarrow X$.

4.11. Theorem. *Let $X$ be an object in $towC$. If $X$ is finitely generated, then $X$ is admissible in $towC$. Consequently, the restriction functor $\mathcal{P}\colon towC/fg \longrightarrow C_{\mathcal{P}cG}$ is a full embedding, where $towC/fg$ denotes the full subcategory of $towC$ determined by finitely generated towers.*

Proof. It is easy to check that $\sum_{finite} cG$ is isomorphic to $cG$. By Proposition 4.8, it follows that $\sum_{finite} cG$ is admissible. Because $X$ is finitely generated, there is an effective epimorphism $\sum_{finite} cG \longrightarrow X$. Now taking into account Proposition 4.9, we obtain that $X$ is also admissible. ∎

4.12. Proposition. *Let $Y = \{\cdots \to Y_2 \longrightarrow Y_1 \longrightarrow Y_0\}$ be an object in $towC$, where the bonding morphisms are denoted by $Y_k^l\colon Y_l \longrightarrow Y_k$, $l \geq k$. If for each $i \geq 0$, there is a finite set $A_i \subset Y_i$ such that for each $n \geq 0$, $\bigcup_{j \geq n} Y_n^j A_j$ generates $Y_n$, then $Y$ is finitely generated.*

Proof. Define $X_n = \sum_{j \geq n} \sum_{A_j} G$ and consider the diagram

$$\begin{array}{ccccc}
\cdots \longrightarrow X_{n+1} & \longrightarrow & X_n \longrightarrow \cdots \\
{\scriptstyle p_{n+1}}\big\downarrow & & \big\downarrow{\scriptstyle p_n} \\
\cdots \longrightarrow Y_{n+1} & \longrightarrow & Y_n \longrightarrow \cdots
\end{array}$$

where the restriction of $p_n$ to $\sum_{A_j} G$ is induced by the map $Y_n^j\colon A_j \to Y_n$. It is clear that $X \cong cG$ and $p\colon X \to Y$ is an epimorphism. Therefore $Y$ is finitely generated. ∎

4.13. Corollary. *1) A tower of finitely generated objects of $C$ is a finitely generated tower.*

*2) A tower of finite objects of $C$ is finitely generated.*

*3) The restricted functors*

$$\mathcal{P}\colon tow(C/fg) \longrightarrow C_{\mathcal{P}cG}$$

$$\mathcal{P}\colon tow(C/f) \longrightarrow C_{\mathcal{P}cG}$$

*are a full embeddings, where $C/fg$ and $C/f$ denote the full subcategories determined by finitely generated objects and finite objects, respectively.*

4.14. Remark. A particular case of Corollary 4.13 are Theorem 3.4 and Corollary 3.5 of [Ch.2].

Next we study some relations between the $\mathcal{P}$ functor and the lim functor.

Recall that for $i \geq 0$, $(cG)_i = \sum\limits_{k \geq i} G_k$, where $G_k$ is a copy of the generator $G$. The identity of $G$ induces a map $G_k \to G_{k+1}$, $k \geq i$. We denote by $sh\colon cG \to cG$ the level map $\{sh_i\colon \sum\limits_{k\geq i} G_k \longrightarrow \sum\limits_{k\geq i} G_k\}$ induced by the maps $G_k \to G_{k+1}$.

Given an object $Y$ in $C_{\mathcal{P}cG}$, we denote by $F_{sh}Y$ the object of $C$ defined by

$$F_{sh}Y = \{y \in Y \mid y \cdot sh = y\}$$

Notice that $F_{sh}$ defines a functor from $C_{\mathcal{P}cG}$ to C.

4.15. Theorem. *The following diagram*

$$\begin{tikzcd} towC \arrow[rr, "\lim"] \arrow[dr, "\mathcal{P}"'] & & C \\ & C_{\mathcal{P}cG} \arrow[ur, "F_{sh}"'] & \end{tikzcd}$$

*is commutative up to natural isomorphism. That is,* $\lim X \cong \{x \in \mathcal{P}X \mid x \cdot sh = x\}$.

Proof. For each $\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})$, there is a map $SH\colon P(X^*\varphi) \longrightarrow P(X^*\varphi)$ which applies $x = (x_{\overline{\varphi}(0)}, x_{\overline{\varphi}(1)}, x_{\overline{\varphi}(2)}, \cdots)$ to the element

$xSH = ((xSH)_{\overline{\varphi}(0)}, (xSH)_{\overline{\varphi}(1)}, \cdots)$,

where for $i \geq 0$, $(xSH)_{\overline{\varphi}(i)} = X^{\overline{\varphi}(i+1)}_{\overline{\varphi}(i)} x_{\overline{\varphi}(i+1)}$. Notice that if $xSH = x$, then $X^{\overline{\varphi}(i+1)}_{\overline{\varphi}(i)} x_{\overline{\varphi}(i+1)} = x_{\overline{\varphi}(i)}$. Therefore $x \in \lim X^*\varphi$.

Associated with the map $X \longrightarrow X^*\varphi$, we have the commutative diagram

$$\begin{tikzcd} \lim X \arrow[r] \arrow[d] & P(X) \arrow[r, shift left, "SH"] \arrow[r, shift right, "id"'] \arrow[d] & P(X) \arrow[d] \\ \lim(X^*\varphi) \arrow[r] & P(X^*\varphi) \arrow[r, shift left, "SH"] \arrow[r, shift right, "id"'] & P(X^*\varphi) \end{tikzcd}$$

where $\lim X$ is the difference kernel of $SH$ and $id$, and similarly for $\lim(X^*\varphi)$. Because $X \longrightarrow X^*\varphi$ is an isomorphism in $towC$ it follows that $\lim X \longrightarrow \lim(X^*\varphi)$ is an isomorphism. Now taking into account that $\operatorname{colim}_\varphi$ preserves difference kernels, we obtain that

$$\lim X \longrightarrow \mathcal{P}X \underset{id}{\overset{sh}{\rightrightarrows}} \mathcal{P}X$$

is a difference kernel. Therefore $\mathrm{lim}X \cong \{x \in \mathcal{P}X \mid x \cdot sh = x\} = F_{sh}\mathcal{P}X$. ∎

Now we can use that $- \otimes^{sh} \mathcal{P}cG\colon C \longrightarrow C_{\mathcal{P}cG}$ is left adjoint to the functor $F_{sh}\colon C_{\mathcal{P}cG} \longrightarrow C$ to obtain the following result:

4.16. Corollary. *The functor* $\lim\colon towC \longrightarrow C$ *can also be represented as follows:*

$$\mathrm{lim}X \cong C_{\mathcal{P}cG}(G \otimes^{sh} \mathcal{P}cG, \mathcal{P}X)$$
$$\mathrm{lim}X \cong C_{\mathcal{P}cG}(\mathcal{P}(conG), \mathcal{P}X)$$

*Moreover, there is a natural map* $G \otimes^{sh} \mathcal{P}cG \longrightarrow \mathcal{P}(conG)$*, where* $conG$ *denotes the level-wise constant tower* $\{\cdots \to G \xrightarrow{id} G\}$*.*

Proof. This follows, because $con\colon C \longrightarrow towC$ is left adjoint to $\lim\colon towC \longrightarrow C$ and $- \otimes^{sh} \mathcal{P}cG\colon \longrightarrow C_{\mathcal{P}cG}$ is left adjoint to $F_{sh}\colon C_{\mathcal{P}cG} \longrightarrow C$, see Proposition 3.9. It is also necessary to take into account the fact that $conG$ is admissible in $towC$. This follows because $conG$ is a tower of finitely generated objects, see Corollary 4.13 and Theorem 4.11. ∎

4.17. Remark. 1) Theorem 4.15 gives a relation between the $\mathcal{P}$ functor and the lim functor for the case of towers. If $X = \{X_i\}$ is a tower, then $\mathcal{P}X \cong \lim\ IX_i$, where $IX_i$ is the reduced countable power. For a more general pro-object $X : J \to C$, Porter [Por.1] uses more general reduced powers to "compute" the lim and $\lim^q$ functors.

2) For a tower of groups $X$, an action of $\mathcal{P}X$ on $\mathcal{P}X$ can be defined by

$$^{x}y = x + y - x \cdot sh \qquad x, y \in \mathcal{P}X.$$

It is easy to check that the space of orbits of this action is isomorphic to the pointed set $\lim^1 X$. The difference of two elements of the same orbit is of the form $x+y-x\cdot sh-y$. Notice that the quotient group obtained by dividing by the normal subgroup generated by the relations $x + y - x \cdot sh - y$, satisfies that the action of $sh$ is trivial and it is an abelian group.

3) For a tower $X$ of abelian groups, we get isomorphisms

$$\lim^1 X \cong \mathrm{Ext}^1(\mathbf{Z}_a \otimes^{sh} \mathcal{P}c\mathbf{Z}_a, \mathcal{P}X)$$
$$\lim^1 X \cong \mathrm{Ext}^1(\mathcal{P}(con\mathbf{Z}_a), \mathcal{P}X)$$

In this case, we also have that $\lim^1 X$ is obtained from $\mathcal{P}X$ by dividing by the subgroup generated by the relations $x - x{\cdot}sh$ for all $x \in \mathcal{P}X$.

4) A global version of Brown's $\mathcal{P}$ functor can be defined for global category $(proC, C)$ (for the definition of $(proC, C)$ see [E-H]). If $X$ is an object in $(proC, C)$, then $\mathcal{P}_g X$ is

defined to be the hom-set $\mathcal{P}_g X = (proC, C)(cG, X)$, where $cG$ is considered as an object in $(proC, C)$, provided with the structure given by the action of $\mathcal{P}_g cG$. We note that for the global version of the $\mathcal{P}$ functor, if $X$ is an object in $(towC, C)$, then

$$U\mathcal{P}_g X \cong \operatorname*{colim}_{\varphi \in In_0(\mathbf{N}_{id}^{\mathbf{N}})} P(X^*\varphi),$$

where $In_0(\mathbf{N}_{id}^{\mathbf{N}}) = \{\varphi \in In(\mathbf{N}_{id}^{\mathbf{N}}) \mid \varphi(0) = 0\}$.

## 5. Applications and properties of the $\mathcal{P}$ functor

In this section, firstly we obtain some consequences of the main Theorems of section 4. We also analyse the structure of the endomorphism set $\mathcal{P}cG$ for the different cases $C = Set, Set_*, Grp, Ab$. Finally, we study some additional properties of the $\mathcal{P}$ functor for the cases $C = Grp, Ab$.

If $C$ is one of the categories: $Set, Set_*, Grp, Ab$, we will denote by $TC$ the corresponding topological category. That is, $TC$ will respectively be one the categories: topological spaces, topological pointed spaces, topological groups or topological abelian groups. We denote by $zcmTC$ the full subcategory of $TC$ determined by zero-dimensional compact metrisable topologies.

Let $X$ be an object in $C$. Consider the set of quotient objects of the form $p: X \longrightarrow F_p$ where $F_p$ is a finite discrete object in $C$. Given two quotients of this form $p: X \longrightarrow F_p$ and $p': X \longrightarrow F_{p'}$, we say that $p \geq p'$ if there is a commutative diagram

$$\begin{array}{ccc} & X & \\ p \swarrow & & \searrow p' \\ F_p & \longrightarrow & F_{p'} \end{array}$$

It is easy to check that $\Lambda = \{p: X \longrightarrow F_p \mid F_p \text{ is a finite discrete quotient object }\}$ with $\geq$, is a directed set. Therefore we can define the functor $TC \longrightarrow pro(C/f): X \longrightarrow \{F_p\}_{p\in\Lambda}$, where $C/f$ denotes the full subcategory of $C$ determined by finite objects. If $X$ has a zero–dimensional compact metrisable topology, then there is a sequence $p_i: X \longrightarrow F_i$ such that $p_{i+1} \geq p_i$ and for any $p$ of $\Lambda$, there is $i \geq 0$ such that $p_i \geq p$. Hence $\{p_i\}_{i\geq 0}$ is cofinal in $\Lambda$, and the tower $\{F_i\}_{i\geq 0}$ is isomorphic to $\{F_p\}_{p\in\Lambda}$.

Consequently it is clear that:

5.1. Theorem. *The category $tow(C/f)$ of towers of finite objects in $C$ is equivalent to the category $zcmTC$ of objects in $TC$ which have a zero–dimensional compact metrisable topology.*

5.2. Remark. T. Porter has pointed out to me that Theorem 5.1 is closely connected with a famous theorem of M.H. Stone [Sto] which gives category equivalences between the category of Boolean spaces, the category of Boolean algebras and the category of Boolean rings.

5.3. Theorem. *There is a full faithful functor* $zcmTC \longrightarrow C_{\mathcal{P}cG}$.

5.4. Remark. 1) There is a full embedding $\varepsilon$ from the proper homotopy category of $\sigma$-compact locally compact Hausdorff spaces into the homotopy category of prospaces considered by Edwards-Hastings [E-H]. If $X$ is a $\sigma$-compact locally compact simplicial complex, then there is a cofinal sequence $\{K_i\}$ of compact subsets of $X$ such that for every $i \geq 0$, $\pi_0(cl(X-K_i))$ is a finite set. Therefore $\pi_0\varepsilon X = \{\pi_0(cl(X-K_i))\}$ is admissible in $towSet$. If $\alpha\colon [0,\infty) \to X$ is a proper ray, then $\alpha$ determines a path-component of $\pi_0(cl(X-K_i))$ and $\pi_0\varepsilon X$ can be considered as an object in $towSet_*$. If $X$ is a simplicial complex as above, we will suppose that $\alpha$ is a simplicial injective map. In this case, the fundamental pro-group can be defined by $\pi_1\varepsilon(X,\alpha) = \{\pi_1(cl(X-K_i)\cup \mathrm{Im}\alpha, \alpha(0))\}$. If $X$ has one Freudenthal end, it is easy to check that $\pi_1\varepsilon(X,\alpha)$ is admissible in $towGrp$. Finally, we also note that for $q \geq 0$, the tower $H_q\varepsilon X = \{H_q(cl(X-K_i))\}$ is admissible in $towAb$, where $H_q$ denotes the singular homology.

2) Let $X$ be a compact metrisable pointed space. Denote by $\check{C}X$ the pro-pointed simplicial set of the Čech nerves associated with the directed set of open coverings of $X$. In this case, it is easy to check that $\pi_0\check{C}X$ is isomorphic to an admissible object in $towSet_*$, $\pi_1\check{C}X$ is isomorphic to an admissible object in $towGrp$, and $H_q\check{C}X$ is isomorphic to an admissible object in $towAb$.

3) As a consequence of Theorems 5.1 and 5.3, for the category of connected locally finite countable simplicial complexes, the following categories are adequate for modelling the proper 0–type. The category of zero-dimensional compact metrisable spaces and the Freudenthal end functor $e$, the category $tow(finite\ sets)$ and the $\pi_0\varepsilon$ functor and $Set_{*\mathcal{P}cS^0}$ and the Brown–Grossman 0–homotopy group $\pi_0^{BG}$. The relations between these functors are given by $e = \lim\, \pi_0\, \varepsilon$, $\quad {\pi_0}^{BG} = \mathcal{P}\ \pi_0\ \varepsilon$. Similarly, for the shape 0-type of compact metrisable spaces, we have the functors $\lim\pi_0\check{C}$, $\pi_0\check{C}$ and $\mathcal{P}\pi_0\check{C}$.

Next we study the different structures of the endomorphism set $\mathcal{P}cG = proC(cG,cG)$ for the different cases $C = Set, Set_*, Grp, Ab$.

1) $C = Set$

The monoid $\mathcal{P}c*$ can be represented as follows: Consider the set $\mathcal{R}_*$ of matrices of the form $A = (A_{ij})$, with $i,j \in \{0,1,2,\cdots\}$, where either $A_{ij} = 0$ or $A_{ij} = 1$, satisfying the following properties:

a) For each $j \geq 0$, the cardinality of $\{i \mid A_{ij} = 1\}$ is 1.

b) For each $i \geq 0$, there exists $j \geq i$ such that for $0 \leq k < i$ and $j \leq l$, $\quad a_{kl} = 0$. Write $\varphi_A(i) = \min\{j \mid j \geq i$ and if $0 \leq k < i$ and $j \leq l$, then $a_{kl} = 0\}$.

Define the equivalence relation $\sim$ by declaring $A \sim A'$ if there exists $j \geq 0$ such that for $j \leq l$ the $l$–column of $A$ agrees with the $l$–column of $A'$; that is, $A$ and $A'$ differ only on a finite number of columns.

Matrix multiplication induces over $\mathcal{R}_*$ a monoid structure that is compatible with the relation $\sim$. Therefore the quotient $\mathcal{R}_*^\infty$ inherits a monoid structure from $\mathcal{R}_*$. If we (also) denote by $\mathbf{N}$ the set of natural numbers provided with the discrete topology, we can consider the monoid $P(\mathbf{N},\mathbf{N})$ of proper maps $\mathbf{N} \to \mathbf{N}$ and the monoid of germs of proper maps $P^\infty(\mathbf{N},\mathbf{N})$. Given a matrix $A$ of $\mathcal{R}_*$, we can define a proper map that will again be denoted by $A\colon \mathbf{N} \longrightarrow \mathbf{N}$ as follows: if $j \geq 0$ the $j$–columm of $A$ has only one element $A_{ij} = 1$, define $A(j) = i$. This gives monoid isomorphisms $\mathcal{R}_* \longrightarrow P(\mathbf{N},\mathbf{N})$ and $\mathcal{R}_*^\infty \longrightarrow P^\infty(\mathbf{N},\mathbf{N})$. The isomorphism

$$\mathcal{R}_*^\infty \longrightarrow \mathcal{P}\ c* = \operatorname*{colim}_{\varphi \in In(\mathbf{N}_{id}^{\mathbf{N}})} p(c*^* \varphi)$$

is given by $A \longrightarrow (A(0), A(1), A(2), \cdots) \in p(c*^* \varphi_A)$.

Given an object $X$ of $towSet$, the action $\mathcal{P}X \times \mathcal{P}c* \longrightarrow \mathcal{P}X$ can be defined as follows: take $x \in p(X^*\varphi)$ and $A \in \mathcal{R}_*$, then $[x][A] = [y]$, where $y \in p((X^*\varphi)^* \varphi_A))$ is defined by

$$y_{\bar\varphi\bar\varphi_A(j)} = X^{+\bar\varphi A(j)}_{\bar\varphi\bar\varphi_A(j)}\ x_{\bar\varphi A(j)} = X^{+\bar\varphi A(j)}_{\bar\varphi\bar\varphi_A(j)}(\sum_l\ x_{\bar\varphi l}\ A_{lj}).$$

2) $C = Set_*$

The monoid $\mathcal{P}cS^0$ can also be represented as a matrix monoid $\mathcal{R}_{S^0}^\infty$ as follows: We consider matrices $A = (A_{ij})$ as above satisfying properties a$'$) and b), where a$'$) is obtained by modifying $a$).

a$'$) For each $j \geq 0$, the cardinality of $\{i \mid A_{ij} = 1\}$ is at most one.

Denote by $\mathbf{N}^* = \mathbf{N} \cup \{*\}$ the Alexandroff compactification of $\mathbf{N}$ by a point $*$ and consider the monoid $Top_*((\mathbf{N}^*, *), (\mathbf{N}^*, *))$ and the monoid $(Top_*)^\infty((\mathbf{N}^*, *), (\mathbf{N}^*, *))$ of germs at $*$ of continuous maps $(\mathbf{N}^*, *) \longrightarrow (\mathbf{N}^*, *)$. Given a matrix $A$, we can define the continuous map $A\colon (\mathbf{N}^*, *) \longrightarrow (\mathbf{N}^*, *)$ such that if $j \geq 0$ and the $j$–column of $A$ has a unique element $A_{ij} = 1$, then $A(j) = i$, otherwise $A(j) = *$. This defines isomorphisms $\mathcal{R}_{S^0} \longrightarrow Top_*(\mathbf{N}^*, *), (\mathbf{N}^*, *))$ and $\mathcal{R}_{S^0}^\infty \longrightarrow (Top_*)^\infty\ ((\mathbf{N}^*, *), (\mathbf{N}^*, *))$. The isomorphism $\mathcal{R}_{S^0}^\infty \longrightarrow \operatorname*{colim}_{\varphi \in In(\mathbf{N}_{id}^{\mathbf{N}})} p(cS^{0*}\varphi)$ is given by $A \longrightarrow (A(0), A(1), A(2), \cdots) \in p(cS^{0*}\varphi_A)$.

Notice that $\mathcal{R}_*^\infty$ is a submonoid of $\mathcal{R}_{S^0}^\infty$.

3) $C = Grp$

Let $F$ be the free group over the countable set of letters $\{x_0, x_1, x_2, \ldots\}$. The multiplication of $F$ will be denoted by $+$, then a typical word of $F$ is of the form $2x_2 + 3x_1 - x_0$, we note that an additive notation does not imply commutativity. Let $\mathcal{R}_{\mathbf{Z}}$ denote the set whose elements are of the form $(w_0, w_1, w_2, \cdots)$, where for $i \geq 0$ $w_i \in F$, satisfying the following property:

For each $i \geq 0$, there exists $j \geq i$ such that $x_0, \cdots, x_{i-1}$ are not letters of the reduction of $w_l$ for $l \geq j$. For a given element $w = (w_0, w_1, w_2, \cdots)$ of $\mathcal{R}_{\mathbf{Z}}$, write

$\varphi_w(i) = \min\{j \mid j \geq i$ and $x_0, \cdots, x_{i-1}$ are not letters of the reduction of $w_l$ for $l \geq j\}$.

The sum is defined by components

$$(w_0, w_1, w_2, \cdots) + (w'_0, w'_1, w'_2, \ldots) = (w_0 + w'_0\ w_1 + w'_1\ w_2 + w'_2, \cdots)$$

Denote by $w(x_{n_1}, \ldots, x_{n_r})$ a word whose reduction has the letters $x_{n_1}, \cdots, x_{n_r}$. The product is defined by substitution as follows

$(w_0, w_1, w_2, \ldots)(w'_0(x_{n^0_1}, \ldots, x_{n^0_{r_0}}), w'_1(x_{n^1_1}, \ldots, x_{n^1_{r_1}}), w'_2(x_{n^2_1}, \ldots, x_{n^2_{r_2}}), \ldots) =$
$= (w'_0(w_{n^0_1}, \ldots, w_{n^0_{r_0}}), w'_1(w_{n^1_1}, \ldots, w_{n^1_{r_1}}), w'_2(w_{n^2_1}, \ldots, w_{n^2_{r_2}}), \ldots)$

It is easy to check that $+$ and $\cdot$ give the structure of a zero–symmetric near–ring to $\mathcal{R}_{\mathbf{Z}}$. The zero element is $(0, 0, 0, \ldots)$ and the unit is represented by $(x_0, x_1, x_2, \ldots)$. Another distinguished element is the shift operator $(x_1, x_2, x_3, \ldots)$ that plays an important role in connection with the inverse limit functor.

Let $I_{\mathbf{Z}}$ be the subset of $\mathcal{R}_{\mathbf{Z}}$ defined by the elements $w = (w_0, w_1, w_2, \ldots)$ such that there exists $m \geq 0$ such that $w_l = 0$ for $l > m$. Then it is easy to check that $I_{\mathbf{Z}}$ is a normal subgroup of $\mathcal{R}_{\mathbf{Z}}$, $(\mathcal{R}_{\mathbf{Z}})(I_{\mathbf{Z}}) \subseteq I_{\mathbf{Z}}$ and $(r + i)s - rs \in I_{\mathbf{Z}}$ for all $i \in I_{\mathbf{Z}}$, $r, s \in \mathcal{R}_{\mathbf{Z}}$. Then $I_{\mathbf{Z}}$ is a ideal of $\mathcal{R}_{\mathbf{Z}}$ and we can consider the quotient near–ring $\mathcal{R}^{\infty}_{\mathbf{Z}} = \mathcal{R}_{\mathbf{Z}}/I_{\mathbf{Z}}$.

For $\mathcal{R}_{\mathbf{Z}}$ and $\mathcal{R}^{\infty}_{\mathbf{Z}}$ we have the near–ring isomorphisms

$$\mathcal{R}_{\mathbf{Z}} \longrightarrow \operatorname*{colim}_{\varphi \in In_0(\mathbf{N}^{\mathbf{N}}_{id})} p(c\mathbf{Z}^*\varphi)$$

$$\mathcal{R}^{\infty}_{\mathbf{Z}} \longrightarrow \operatorname*{colim}_{\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})} p(c\mathbf{Z}^*\varphi)$$

defined by $w = (w_0, w_1, w_2, \ldots) \longrightarrow [(w_0, w_1, w_2, \ldots)]$, where $(w_0, w_1, w_2, \ldots) \in p(c\mathbf{Z}^*\varphi_{\mathbf{w}})$ and $\varphi_w$ is the map defined above.

4) $C = Ab$

Let $\mathcal{R}_{\mathbf{Z}_a}$ denote the ring of integer matrices $A = (a_{ij})$ where i and j are non negative integers and each row and each column have finitely many non zero elements.

If $A$ is a matrix of $\mathcal{R}_{\mathbf{Z}_a}$ for each $i \geq 0$, there exists $j \geq i$ such that $a_{kl} = 0$ for $l \geq j$ and $k < i$. For a given matrix $A$, write $\varphi_A(i) = \min\{j \mid j \geq i$ and if $k < i$ and $l \geq j$, then $a_{kl} = 0\ \}$. Let $I_{\mathbf{Z}_a}$ be the subset of $\mathcal{R}_{\mathbf{Z}_a}$ defined by the finite matrices. Then it is easy to check that $I_{\mathbf{Z}_a}$ is an ideal of $\mathcal{R}_{\mathbf{Z}_a}$ and we can consider the quotient ring $\mathcal{R}^{\infty}_{\mathbf{Z}_a} = \mathcal{R}_{\mathbf{Z}_a}/I_{\mathbf{Z}_a}$.

We also have the canonical ring isomorphisms:

$$\mathcal{R}_{\mathbf{Z}_a} \longrightarrow \operatorname*{colim}_{\varphi \in In_0(\mathbf{N}^{\mathbf{N}}_{id})} p(c{\mathbf{Z}_a}^*\varphi)$$
$$\mathcal{R}^{\infty}_{\mathbf{Z}_a} \longrightarrow \operatorname*{colim}_{\varphi \in In(\mathbf{N}^{\mathbf{N}}_{id})} p(c{\mathbf{Z}_a}^*\varphi)$$

defined by

$A \longrightarrow [($0-column of $A$, 1-column of $A$, 2-column of $A, \ldots)]$,

where (0-column of $A$, 1-column of $A$, 2-column of $A, \ldots) \in p(c{\mathbf{Z}_a}^*\varphi_A)$ and $\varphi_A$ is the map defined above.

Let $F^a$ be the free abelian group generated by the countable set $\{x_0, x_1, x_2, \ldots\}$. That is, $F^a = f_a\{x_0, x_1, x_2, \ldots\}$, where $f_a\colon Set \longrightarrow Ab$ denotes the free abelian functor. Consider the following sequence of subgroups : $F_0^a = F^a$, $F_1^a = f_a\{x_1, x_2, \ldots\}$, $F_2^a = f_a\{x_2, x_3, \ldots\}$, etc. This family of subgroups defines on $F^a$ the structure of a topological abelian group. Denote by $TAb$ the category of topological abelian groups.

Let $End_{TAb}(F^a, F^a)$ denote the ring of continuous endomorphisms of $F^a$. If $f\colon F^a \longrightarrow$ $\longrightarrow F^a$ is a continuous homomorphism, because $x_i \longrightarrow 0$, we have that $w_i = fx_i \longrightarrow 0$. This implies that we have a canonical isomorphism $R_{\mathbf{Z}_a} \longrightarrow End_{TAb}(F^a, F^a)$.

Given two continuous homomorphisms $f, g\colon F^a \longrightarrow F^a$ we say that $f$ and $g$ have the same germ if there exists $n_0$ such that for every $n \geq n_0 \quad f(x_n) = g(x_n)$. Let $End^{\infty}_{TAb}(F^a, F^a)$ denote the ring of selfgerms of $F^a$, it is also clear that $R^{\infty}_{\mathbf{Z}_a}$ is isomorphic to $End^{\infty}_{TAb}(F^a, F^a)$.

Next we compare the different $\mathcal{P}$ functors for the cases $C = Set_*, Grp, Ab$.

5.5. Proposition. *Consider the diagram*

$$\begin{tikzcd} towGps \arrow[r, "\mathcal{P}"] & Gps_{\mathcal{P}c\mathbf{Z}} \\ towSet_* \arrow[u, "f"] \arrow[r, "\mathcal{P}"'] & Set_{*\mathcal{P}cS^0} \arrow[u, "f"'] \end{tikzcd}$$

*where the functor $f\colon towSet_* \to towGps$ is induced by the free functor $Set_* \to Gps$ and $f\colon Set_{*\mathcal{P}cS^0} \to Gps_{\mathcal{P}c\mathbf{Z}}$ is the free functor associated with the algebraic "forgetful" functor $Gps_{\mathcal{P}c\mathbf{Z}} \to Set_{*\mathcal{P}cS^0}$ (see [Pa; th 1 of 3.4]). Then the unit $X \longrightarrow ufX$ (of the pair of adjoint functors: $f\colon towSet_* \longrightarrow towGps$, $\quad u\colon towGps \longrightarrow towSet_*$) induces a natural and epimorphic transformation $\eta_X\colon f\mathcal{P}X \longrightarrow \mathcal{P}fX$.*

Proof. The unit transformation $Y \longrightarrow ufY$ induces the transformation $\mathcal{P}Y \underset{\eta}{\longrightarrow} \mathcal{P}ufY = u\mathcal{P}fY$. By adjointness we obtain the desired transformation $f\mathcal{P}Y \longrightarrow \mathcal{P}fY$.

An element of $\mathcal{P}fY$ can be represented as a sequence of words:

$$a = [((\varepsilon_0 y_0 + \ldots + \varepsilon_{r_0} y_{r_0})_{\bar{\varphi}(0)}, (\varepsilon_{r_0+1} y_{r_0+1} + \ldots + \varepsilon_{r_1} y_{r_1})_{\bar{\varphi}(1)}, \ldots)]$$

where $\varepsilon_k \in \{-1, 0, 1\}$, $\quad y_0, \ldots, y_{r_0} \in Y^+_{\bar{\varphi}(0)}$, $\quad y_{r_0+1}, \ldots, y_{r_1} \in Y^+_{\bar{\varphi}(1)}$, etc.

If you take, one by one, the "letters" of these words, you obtain an element of $f\mathcal{P}Y$

$$b = [(y_0, \ldots, y_{r_0}, y_{r_0+1}, \ldots, y_{r_1}, y_{r_1+1}, \ldots)].$$

If you replace the $y$'s of $a$ by $x$'s, you will have an element of $\mathcal{P}c\mathbf{Z}$

$$w = [((\varepsilon_0 x_0 + \ldots + \varepsilon_{r_0} x_{r_0})_{\bar{\varphi}(0)}, (\varepsilon_{r_0+1} x_{r_0+1} + \ldots + \varepsilon_{r_1} x_{r_1})_{\bar{\varphi}(1)}, \ldots)]$$

It is clear that $(\eta b)w = a$, then $\eta(bw) = a$. Therefore $\eta_X\colon f\mathcal{P}Y \longrightarrow \mathcal{P}fY$ is a surjective map. ∎

5.6. Corollary. *Consider the functor $f\colon towSet_* \longrightarrow towGps$. If $X$ is admissible in $towSet_*$, then $fX$ is admissible in $towGrp$.*

Proof. We use the facts that $\mathcal{P}$ is a faithful functor and $f\mathcal{P}X \longrightarrow \mathcal{P}fX$ is an epimorphism to obtain:

$$\begin{aligned} &towGps(fX,Y) \subset Gps_{\mathcal{P}c\mathbf{Z}}(\mathcal{P}fX,\mathcal{P}Y) \subset Gps_{\mathcal{P}c\mathbf{Z}}(f\mathcal{P}X,\mathcal{P}Y) = \\ &= Set_{*\mathcal{P}cS^0}(\mathcal{P}X,u\mathcal{P}Y) = Set_{*\mathcal{P}cS^0}(\mathcal{P}X,\mathcal{P}uY) = \\ &= towSet_*(X,uY) = towGps(fX,Y). \end{aligned}$$

where "$\subset$" denotes an injective map and "$=$" denotes an isomorphism. Because the composite is the identity, we have $towGps(fX,Y) \cong Gps_{\mathcal{P}c\mathbf{Z}}(\mathcal{P}fX,\mathcal{P}Y)$. ∎

We include here some additional properties of the $\mathcal{P}$ functor for the category of towers of abelian groups.

5.7. Proposition. *The functor $\mathcal{P}\colon towAb \longrightarrow Ab_{\mathcal{P}c\mathbf{Z}_a}$ preserves finite colimits.*

Proof. In an abelian category the product and coproduct of $X$ and $Y$ are both given by an object $Z$ and morphisms $i\colon X \to Z$, $j\colon Y \to Z$, $p\colon Z \to X$ and $q\colon Z \to Y$ such that $pi = id$, $qj = id$, $qi = 0$, $pj = 0$ and $ip + jq = id$. Since $towAb$ is an abelian category, see [A–M], and $\mathcal{P}$ is an additive functor, it follows that $\mathcal{P}$ preserves finite coproducts. Given a morphism $f\colon X \to Y$ in $towAb$, $f$ factorizes as $X \xrightarrow{g} X' \xrightarrow{k} Y$, where $g$ is an epimorphism and $k$ is a monomorphism. It is easy to check that coker $f \cong$ coker $k$. By Remark 1) after Theorem 4.4 and by Proposition 4.6, we have that $\mathcal{P}$ preserves monomorphisms and epimorphisms, so we also obtain that coker $\mathcal{P}f \cong$ coker $\mathcal{P}k$. Because $U\mathcal{P} \cong towAb(c\mathbf{Z}_a, -)$, we have the exact sequence:

$$U\mathcal{P}X' \longrightarrow U\mathcal{P}Y \longrightarrow U\mathcal{P}\text{coker } k \longrightarrow \mathrm{Ext}_{towAb}(c\mathbf{Z}_a, X')$$

Since $c\mathbf{Z}_a$ is a projective object, see [He.1], we have that $\mathrm{Ext}_{towAb}(c\mathbf{Z}_a, X') \cong 0$. Therefore $0 \longrightarrow U\mathcal{P}X' \longrightarrow U\mathcal{P}Y \longrightarrow U\mathcal{P}\text{coker } k \longrightarrow 0$ is a short exact sequence. Since $U$ reflects monomorphisms, epimorphisms and kernels, we also have that $0 \longrightarrow \mathcal{P}X' \longrightarrow \mathcal{P}Y \longrightarrow \mathcal{P}\text{coker } k \longrightarrow 0$ is a short exact sequence. Then $\mathcal{P}\text{coker } f \cong \mathcal{P}\text{coker } k \cong$ coker $\mathcal{P}k \cong$ coker $\mathcal{P}f$. ∎

We next consider the inclusion functor $i\colon Ab \longrightarrow Gps$ and the abelianization functor $a\colon Gps \longrightarrow Ab$ which is the left adjoint of $i$; that is, $Ab(aX,Y) \cong Gps(X,iY)$. We shall also consider the unitary near–ring epimorphism $\mathcal{P}c\mathbf{Z} \longrightarrow \mathcal{P}c\mathbf{Z}_a$ that induces an inclusion functor $i\colon Ab_{\mathcal{P}c\mathbf{Z}_a} \longrightarrow Gps_{\mathcal{P}c\mathbf{Z}}$ which has a left adjoint $d\colon Gps_{\mathcal{P}c\mathbf{Z}} \longrightarrow Ab_{\mathcal{P}c\mathbf{Z}_a}$. It is easy to check that the diagram

$$\begin{tikzcd} towAb \arrow[r, "\mathcal{P}"] \arrow[d, "i"'] & Ab_{\mathcal{P}c\mathbf{Z}_a} \arrow[d, "i"'] \\ towGps \arrow[r, "\mathcal{P}"'] & Gps_{\mathcal{P}c\mathbf{Z}} \end{tikzcd}$$

is commutative up to natural isomorphism. The following result proves that $\mathcal{P}a = d\mathcal{P}$.

5.8. Theorem. *Consider the diagram*

$$\begin{tikzcd} towAb \arrow[r, "\mathcal{P}"] & Ab_{\mathcal{P}c\mathbf{Z}_a} \arrow[r, "U"] & Ab \\ towGps \arrow[u, "a"] \arrow[r, "\mathcal{P}"'] & Gps_{\mathcal{P}c\mathbf{Z}} \arrow[u, "d"'] \arrow[r, "U"'] & Gps \arrow[u, "a"'] \end{tikzcd}$$

*where $a$ and $d$ are left adjoint to the corresponding inclusion functors. Then*

*1) There is a natural equivalence $d\mathcal{P}X \longrightarrow \mathcal{P}aX$ induced by the unit transformation $X \longrightarrow iaX$ ($d\mathcal{P}X \longrightarrow d\mathcal{P}iaX \cong di\mathcal{P}aX \cong \mathcal{P}aX$).*

*2) The natural transformation $aUY \longrightarrow UdY$ is epimorphic.*

Proof. Given an object $X$ in *towGps*, consider the following diagram, where several notational abuses are made in order to have a shorter notation:

$$\begin{tikzcd} \mathcal{P}[X,X] \arrow[r] & \mathcal{P}X \arrow[r] & \mathcal{P}aX \\ \cup & & \\ D\mathcal{P}X \arrow[r] & \mathcal{P}X \arrow[u, "id"'] \arrow[r] & d\mathcal{P}X \arrow[u] \\ \cup & & \\ {[\mathcal{P}X,\mathcal{P}X]} \arrow[r] & \mathcal{P}X \arrow[u, "id"'] \arrow[r] & a\mathcal{P}X \arrow[u] \end{tikzcd}$$

where if $X = \{X_i\}, [X,X] = \{[X_i, X_i]\}$ and $[\ ,\ ]$ denotes the normal subgroup generated by the commutators $[x,y] = x+y-x-y$. By (4.1), Remark 1) after Theorem 4.4 and

(4.4), the first row of the diagram above is exact. In the second row $D\mathcal{P}X$ is the sub $\mathcal{P}c\mathbf{Z}$–group generated by $xw+yw-(x+y)w$ where $x, y$ are elements of $\mathcal{P}X$ and $w \in \mathcal{P}c\mathbf{Z}$. Notice that if $w = -1$, we have $-x-y+x+y$. Therefore $D\mathcal{P}X$ contains the commutator subgroup $[\mathcal{P}X, \mathcal{P}Y]$. Recall that if $H$ is a group and $y \in [H, H]$ then $y = \varepsilon_1\, y_1 + \cdots + \varepsilon_r\, y_r$ where $\varepsilon_i \in \{-1, 0, 1\}$ and $y_i = [a_i, b_i]$ with $a_i, b_i \in H$.

An element $a$ of $\mathcal{P}[X, X]$ can be represented by a sequence of words

$a = [((\varepsilon_0\; y_0 + \cdots + \varepsilon_{r_0}\; y_{r_0})_{\overline{\varphi}(0)}, (\varepsilon_{r_0+1}\; y_{r_0+1} + \cdots + \varepsilon_{r_1}\; y_{r_1})_{\overline{\varphi}(1)}, \cdots)]$

where $\varepsilon_k \in \{-1, 0, 1\}$, $y_0, \cdots, y_{r_0}$ are basic commutators of $X^+_{\overline{\varphi}(0)}$, $y_{r_0+1}, \cdots, y_{r_1}$ are basic commutators of $X^+_{\overline{\varphi}(1)}$, etc.

If you take, one by one, the basic commutators of these words, you obtain an element of $[\mathcal{P}X, \mathcal{P}X]$

$b = [(y_0, \cdots, y_{r_0}, y_{r_0+1}, \cdots, y_{r_1}, y_{r_1+1}, \cdots)]$

and by replacing the $y$'s of $a$ by $x$'s, we get an element of $\mathcal{P}c\mathbf{Z}$,

$w = [((\varepsilon_0\; x_0 + \cdots + \varepsilon_{r_0}\; x_{r_0})_{\overline{\varphi}(0)},\; (\varepsilon_{r_0+1}\; x_{r_0+1} + \cdots + \varepsilon_{r_1}\; x_{r_1})_{\overline{\varphi}(1)}, \cdots)]$

satisfying $bw = a$. Since $D\mathcal{P}X$ is a sub–$\mathcal{P}c\mathbf{Z}$–group, $a \in D(\mathcal{P}X)$. Therefore $D\mathcal{P}X = \mathcal{P}[X, X]$ and this implies that $\mathcal{P}aX = d\mathcal{P}X$. ∎

## 6. The left adjoint for the $\mathcal{P}$ functor.

In this section, we construct a left adjoint functor for the $\mathcal{P}$ functor.

First we introduce some notation and a technical result (Proposition 6.1) that gives the construction of the left adjoint. Applying this proposition to the $\mathcal{P}$ functor, we have the desired result.

Assume that $\mathcal{A}, \mathcal{B}$ are categories with infinite sums, $\mathcal{P}\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a given functor and $H$ is an object of $\mathcal{A}$ such that for any $X$ of $\mathcal{A}$

$$\mathcal{P}\colon \mathcal{A}(H, X) \longrightarrow \mathcal{B}(\mathcal{P}H, \mathcal{P}X)$$

is a bijection.

Let $\mathcal{S}$ be the category whose objects are objects of $\mathcal{B}$ with a given decomposition of the form $\sum_{\alpha\in A} \mathcal{P}H_\alpha$, where $A$ is an index set, $\Sigma$ denotes the sum or coproduct in $\mathcal{B}$ and $H_\alpha = H$ for all $\alpha \in A$. The morphism–set from $\sum_{\alpha\in A} \mathcal{P}H_\alpha$ to $\sum_{\beta\in B} \mathcal{P}H_\beta$ is given by $\mathcal{B}(\sum_{\alpha\in A} \mathcal{P}H_\alpha, \sum_{\beta\in B} \mathcal{P}H_\beta)$

Let $in_{H_\beta}\colon H_\beta \longrightarrow \sum_{\beta\in B} H_\beta$ denote the canonical "inclusion" into the coproduct, where it is assumed that $H_\beta = H$ for any $\beta \in B$. Applying the functor $\mathcal{P}$ and the universal property of the sum we have the morphisms:

$$\mathcal{P}in_{H_\beta}\colon \mathcal{P}H_\beta \longrightarrow \mathcal{P}(\sum_{\beta\in B} H_\beta)$$

$$\sum_{\beta\in B} \mathcal{P}in_{H_\beta}\colon \sum_{\beta\in B} \mathcal{P}H_\beta \longrightarrow \mathcal{P}(\sum_{\beta\in B} H_\beta)$$

Next we are going to construct a functor $l: \mathcal{S} \longrightarrow \mathcal{A}$. Given an object $\sum_{\alpha\in A} \mathcal{P}H_\alpha$ of $\mathcal{S}$, define

$$l(\sum_{\alpha\in A} \mathcal{P}H_\alpha) = \sum_{\alpha\in A} H_\alpha$$

where $H_\alpha = H$ for any $\alpha \in A$.

If $u: \sum_{\alpha\in A} \mathcal{P}H_\alpha \longrightarrow \sum_{\beta\in B} \mathcal{P}H_\beta$ is a morphism of $\mathcal{S}$, then $u = \sum_{\alpha\in A} u_\alpha$, where $u_\alpha = u\ in_{\mathcal{P}H_\alpha}$ and $in_{\mathcal{P}H_\alpha}: \mathcal{P}H_\alpha \longrightarrow \sum_{\alpha\in A} \mathcal{P}H_\alpha$ are the canonical "inclusions".

For each $\alpha \in A$, consider the composition

$$\mathcal{P}H_\alpha \xrightarrow{u_\alpha} \sum_{\beta\in B} \mathcal{P}H_\beta \xrightarrow{\sum_{\beta\in B} \mathcal{P}in_{H_\beta}} \mathcal{P}(\sum_{\beta\in B} H_\beta).$$

Since $\mathcal{P}: \mathcal{A}(H_\alpha, \sum_{\beta\in B} H_\beta) \longrightarrow \mathcal{B}(\mathcal{P}H_\alpha, \mathcal{P}(\sum_{\beta\in B} H_\beta))$ is a bijection, there is a unique $lu_\alpha: H_\alpha \longrightarrow \sum_{\beta\in B} H_\beta$ such that $\mathcal{P}lu_\alpha = (\sum_{\beta\in B} \mathcal{P}in_{H_\beta})u_\alpha$. Then define

$$lu = \sum_{\alpha\in A} lu_\alpha$$

Next, we check that $l: \mathcal{S} \longrightarrow \mathcal{A}$ is a functor. We start by showing that $l$ preserves identities.

The canonical "inclusions" $in_{H_\alpha}: H_\alpha \longrightarrow \sum_{\alpha\in A} H_\alpha$ are such that the diagram

$$\begin{tikzcd}
 & \mathcal{P}(\sum_{\alpha\in A} H_\alpha) \\
\mathcal{P}H_\alpha \arrow[ur, "\mathcal{P}in_{H_\alpha}"] \arrow[r, "in_{\mathcal{P}H_\alpha}"'] & \sum_{\alpha\in A} \mathcal{P}H_\alpha \arrow[u, "\sum_{\alpha\in A} \mathcal{P}in_{H_\alpha}"']
\end{tikzcd}$$

is commutative, therefore

$$l(id) = l(\sum_{\alpha\in A} in_{\mathcal{P}H_\alpha}) = \sum_{\alpha\in A} l\ in_{\mathcal{P}H_\alpha} = \sum_{\alpha\in A} in_{H_\alpha} = id.$$

Given two morphisms

$$\sum_{\alpha\in A} \mathcal{P}H_\alpha \xrightarrow{u} \sum_{\beta\in B} \mathcal{P}H_\beta \xrightarrow{v} \sum_{\gamma\in C} \mathcal{P}H_\gamma$$

we have the commutative diagram

$$\begin{tikzcd}
 & \mathcal{P}(\sum\limits_{\beta\in B} H_\beta) \arrow[r, "\mathcal{P}(\Sigma l v_\beta)"] & \mathcal{P}(\sum\limits_{\gamma\in C} H_\gamma) \\
\mathcal{P}H_\alpha \arrow[r, "u_\alpha"'] \arrow[ur, "\mathcal{P}lu_\alpha"] & \sum\limits_{\beta\in B} \mathcal{P}H_\beta \arrow[u, "\Sigma\mathcal{P}in_{H_\beta}"] \arrow[r, "v=\Sigma v_\beta"'] \arrow[ur, "\Sigma\mathcal{P}lv_\beta"] & \sum\limits_{\gamma\in C} \mathcal{P}H_\gamma \arrow[u, "\Sigma\mathcal{P}in_{H_\gamma}"']
\end{tikzcd}$$

Therefore $l(vu_\alpha) = (\sum\limits_{\beta\in B} lv_\beta)lu_\alpha$. Then we have:

$l(vu) = l(\sum\limits_{\alpha\in A}(vu)_\alpha) = l(\sum\limits_{\alpha\in A} vu_\alpha) = \sum\limits_{\alpha\in A} l(vu_\alpha) = \sum\limits_{\alpha\in A}(\sum\limits_{\beta\in B} lv_\beta)lu_\alpha =$

$= \sum\limits_{\alpha\in A} lv\ lu_\alpha = lv\ (\sum\limits_{\alpha\in A} lu_\alpha) = (lv)(lu).$

This implies that $l: \mathcal{S} \longrightarrow \mathcal{A}$ is a functor.

The following properties of $l$ will also be used

a) The transformation

$$\mathcal{A}(l(\sum_{\alpha\in A}\mathcal{P}H_\alpha), Y) \longrightarrow \mathcal{B}(\sum_{\alpha\in A}\mathcal{P}H_\alpha, \mathcal{P}Y)$$

given by $f = \sum\limits_{\alpha\in A} f_\alpha \longrightarrow \sum\limits_{\alpha\in A}\mathcal{P}f_\alpha$ is a bijection.

b) Given morphisms $u: \sum\limits_{\alpha\in A}\mathcal{P}H_\alpha \longrightarrow \sum\limits_{\beta\in B}\mathcal{P}H_\beta$ and $g: Y \longrightarrow Y'$, the following diagram is commutative

$$\begin{tikzcd}
\mathcal{A}(l(\sum\limits_{\alpha\in A}\mathcal{P}H_\alpha), Y') \arrow[r] & \mathcal{B}(\sum\limits_{\alpha\in A}\mathcal{P}H_\alpha, \mathcal{P}Y') \\
\mathcal{A}(l(\sum\limits_{\beta\in B}\mathcal{P}H_\beta), Y) \arrow[r] \arrow[u] & \mathcal{B}(\sum\limits_{\beta\in B}\mathcal{P}H_\alpha, \mathcal{P}Y) \arrow[u]
\end{tikzcd}$$

that is, for a given $f: l(\sum\limits_{\beta\in B}\mathcal{P}H_\beta) \longrightarrow Y$, we have

$$\sum_{\alpha\in A}\mathcal{P}(gfu)_\alpha = \mathcal{P}g(\sum_{\beta\in B}\mathcal{P}f_\beta)u$$

Using this notation and the properties of $l: \mathcal{S} \longrightarrow \mathcal{A}$, we can prove:

6.1. Proposition. *Suppose that $\mathcal{A}, \mathcal{B}$ are two categories with infinite sums and difference cokernels and $\mathcal{P}: \mathcal{A} \to \mathcal{B}$ a functor. Assume that we have:*

*a) An object $H$ of $\mathcal{A}$ such that for any $X$ of $\mathcal{A}$, the map*

$$\mathcal{P}: \mathcal{A}(H, X) \longrightarrow \mathcal{B}(\mathcal{P}H, \mathcal{P}X): f \longrightarrow \mathcal{P}f$$

*is a bijection.*

*b) Two functors $F_1, F_0: \mathcal{B} \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is the category defined above, and two natural transformations $u, v: F_1 \longrightarrow F_0$ such that the functor difcoker$(u, v): \mathcal{B} \longrightarrow \mathcal{B}$ defined by*

$$\text{difcoker}(u,v)B = \text{difcoker}(F_1B \underset{v_B}{\overset{u_B}{\rightrightarrows}} F_0B)$$

*is equivalent to the identity functor of $\mathcal{B}$.*

*Then the functor $\mathcal{P}: \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint $\mathcal{L}: \mathcal{B} \longrightarrow \mathcal{A}$.*

Proof. By considering the functor $l: \mathcal{S} \longrightarrow \mathcal{B}$ defined above, we define $\mathcal{L}: \mathcal{B} \longrightarrow \mathcal{A}$ by

$$\mathcal{L}B = \text{difcoker}(lF_1B \underset{lv_B}{\overset{lu_B}{\rightrightarrows}} lF_0\ B).$$

Now we have

$$\begin{aligned}
\mathcal{A}(\mathcal{L}B, A) &= \mathcal{A}(\text{difcoker}(lF_1B \underset{lv_B}{\overset{lu_B}{\rightrightarrows}} lF_0B), A) \\
&\cong \text{difker}(\mathcal{A}(lF_0B, A) \underset{(lv_B)^*}{\overset{(lu_B)^*}{\rightrightarrows}} \mathcal{A}(lF_1B, A)) \\
&\cong \text{difker}(\mathcal{B}(F_0B, \mathcal{P}A) \underset{v_B^*}{\overset{u_B^*}{\rightrightarrows}} \mathcal{B}(F_1B, \mathcal{P}A)) \\
&\cong \mathcal{B}(\text{difcoker}(F_1B \underset{v_B}{\overset{u_B}{\rightrightarrows}} F_0B), \mathcal{P}A) \\
&\cong \mathcal{B}(B, \mathcal{P}A).
\end{aligned}$$

■

To apply Proposition 6.1, we need to have a category with infinite sums and difference cokernels. The category *proC* has difference cokernels and the following Lemma shows that it also has infinite sums.

6.2. Lemma. *If $C$ has infinite coproducts then proC is also provided with infinite coproducts.*

PROOF. Suppose we have, for each $i \in I$, a pro-object $X_i\colon J_i \longrightarrow C$. Consider the left filtering small category $\prod_{i\in I} J_i$ and define $\sum_{i\in I} X_i\colon \prod_{i\in I} J_i \longrightarrow C$ by $\sum_{i\in I} X_i \ ((j_i)_{i\in I}) = = \sum_{i\in I} X_i(j_i)$, $((j_i)_{i\in I}) \in \prod_{i\in I} J_i$. Associated with the projections $p_i\colon \prod_{i\in I} J_i \longrightarrow J_i$ we have the maps $in_i(p_i((j_i)_{i\in I}))\colon X_i(j_i) \to \sum_{i\in I} X(j_i)$ that define the "inclusions" $in_i\colon X_i \longrightarrow \sum_{i\in I} X_i$. It is easy to check that $\sum_{i\in I} X_i$ verifies the universal property of the coproduct in *proC*. ∎

Recall that $C$ denotes one of the categories: *Set*, $Set_*$, *Grp*, *Ab*, and $C_{\mathcal{P}cG}$ respectively denotes one of the categories:$Set_{\mathcal{P}c*}$, $Set_{*\mathcal{P}cS^0}$, $Gps_{\mathcal{P}cZ}$, $Ab_{\mathcal{P}c\mathbf{Z}_a}$.

To shorten notation, the composition of forgetful functors $C_{\mathcal{P}cG} \xrightarrow{U} C \xrightarrow{u} Set$ will be denoted by $v = uU\colon C_{\mathcal{P}cG} \longrightarrow Set$ and the composition of free functors $Set \xrightarrow{f} C \xrightarrow{F} C_{\mathcal{P}cG}$ by $g = Ff\colon Set \longrightarrow C_{\mathcal{P}cG}$.

The main result of this section is the following:

6.3. THEOREM. *The functor $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ has a left adjoint $\mathcal{L}\colon C_{\mathcal{P}cG} \longrightarrow proC$.*

PROOF. We are going to check that the conditions of Proposition 6.1 are satisfied. Take $\mathcal{A} = proC$ and $\mathcal{B} = C_{\mathcal{P}cG}$.

a) By Proposition 4.8, the object $H = cG$ is admissible. Then for any $X$ in *proC*,

$$proC(cG, X) \cong C_{\mathcal{P}cG}(\mathcal{P}cG, \mathcal{P}X).$$

In the cases we are considering for any object $B$ of $C_{\mathcal{P}cG}$, the natural transformation $p_B\colon gvB \longrightarrow B$ is a (surjective) epimorphism. By Lemma 3 and Corollary 4 of section 3.4 of [Pa], we have that if we consider the fibre product

$$\begin{tikzcd} (gvB \times_B gvB) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & gvB \arrow[d, "p_B"] \\ gvB \arrow[r, "p_B"'] & B \end{tikzcd}$$

then

$$gvB \times_B gvB \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} gvB \xrightarrow[p_B]{} B$$

is a difference cokernel. Since $gv(gvB \times_B gvB) \longrightarrow gvB \times_B gvB$ is an epimorphism, it also follows that

$$gv(gvB \times_B gvB) \overset{pr_1 p}{\underset{pr_2 p}{\rightrightarrows}} gvB \xrightarrow[p_B]{} B$$

is also a difference cokernel.

Then we can define the functors $F_0, F_1 \colon C_{\mathcal{P}cG} \longrightarrow \mathcal{S}$ by

$$F_0B = gvB$$
$$F_1B = gv(gvB \underset{B}{\times} gvB)$$

and the natural transformations by $u = pr_1\ p$ and $v = pr_2\ p$.

Now we are under the conditions of Proposition 6.1, to obtain that $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ has a left adjoint $\mathcal{L}\colon C_{\mathcal{P}cG} \longrightarrow proC$. ∎

6.4. Corollary. *The functors $\mathcal{P}\colon proSet \longrightarrow Set_{\mathcal{P}c*}$, $\mathcal{P}\colon proSet_* \longrightarrow Set_{*\mathcal{P}cS^0}$, $\mathcal{P}\colon proGps \longrightarrow Gps_{\mathcal{P}c\mathbf{Z}}$ and $\mathcal{P}\colon proAb \longrightarrow Ab_{\mathcal{P}c\mathbf{Z}a}$ have left adjoints.*

6.5. Remark. M.I.C. Beattie [Be] has constructed an equivalence between the category of finitely presented towers of abelian groups and finitely presented $\mathcal{P}c\mathbf{Z}_a$–abelian groups or $\mathcal{P}c\mathbf{Z}_a$–modules. This equivalence is also given by the restrictions of the functor $\mathcal{P}\colon pro\mathcal{A}b \longrightarrow \mathcal{A}b_{\mathcal{P}c\mathbf{Z}_a}$ and its left adjoint $\mathcal{L}\colon \mathcal{A}b_{\mathcal{P}c\mathbf{Z}_a} \longrightarrow pro\mathcal{A}b$ to the corresponding full subcategories.

## 7. Global towers and topological abelian groups.

In this section we analyse some relations between towers of abelian groups and topological abelian groups. As a consequence of these relations, we prove that the categories of towers and global towers of abelian groups do not have countable sums. This implies that neither category is equivalent to a category of modules.

7.1. Definition. *Let $N$ be a neighbourhood of the zero element $0$ of a topological group $B$. We will say that $N$ is structured if $N$ is also a subgroup of $B$. A topological abelian group is said to be locally structured if it has a neighbourhood base at $0$ of structured neighbourhoods.*

Let $TAb$ denote the category of topological abelian groups and $STAb$ the full subcategory determined by locally structured topological abelian groups.

Next we define two functors $L\colon (proAb, Ab) \longrightarrow STAb$ and $N\colon STAb \longrightarrow (proAb, Ab)$ such that $L$ is left adjoint to $N$.

Recall that an object $X$ of $(proAb, Ab)$ is a morphism $X = ({}^{\infty}X \longrightarrow X_0)$ where ${}^{\infty}X$ is an object of $proAb$ and $X_0$ is an object of the category $Ab$ which can be considered as a full subcategory of $proAb$. A morphism $f\colon X \longrightarrow Y$ in $(proAb, Ab)$ consists of a pair of morphisms $f = ({}^{\infty}f, f_0)$ such that the following diagram

$$\begin{tikzcd} {}^{\infty}X \arrow[r, "{}^{\infty}f"] \arrow[d] & {}^{\infty}Y \arrow[d] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

is commutative in *proAb*.

Any object of $(proAb, Ab)$ can be represented up to isomorphism by a functor $X{:}\,\Lambda \longrightarrow Ab$ where $\Lambda$ is a directed set with a final element $0 \quad (\lambda \geq 0, \forall\lambda)$. If $\lambda \geq \mu$, let $X^\lambda_\mu{:}\,X_\lambda \longrightarrow X_\mu$ denote the corresponding bonding morphism. Associated with $X$ we have ${}^\infty X = X{:}\,\Lambda \longrightarrow Ab$ which is an object of *proAb*, $X_0$ which is an object of *Ab* (or a constant pro–abelian group) and the natural morphism ${}^\infty X \longrightarrow X_0$ given by the identity ${}^\infty X_0 \longrightarrow X_0$.

Next we use this notation to define a functor $L{:}\,(proAb, Ab) \longrightarrow STAb$. Given an object $X$ of $(proAb, Ab)$, $LX$ is defined to be the abelian group $X_0$ together with the locally structured topology defined by the subgroups $\mathrm{Im}X^\lambda_0$, where $X^\lambda_0{:}\,X_\lambda \longrightarrow X_0$ are bonding maps of $X$. Notice that given $\lambda, \mu$ there exists $\gamma$ such that $\mathrm{Im}X^\gamma_0 \subset (\mathrm{Im}X^\lambda_0) \cap (\mathrm{Im}X^\mu_o)$. This implies that the neighbourhood local base $\{\mathrm{Im}X^\lambda_0\}$ defines a topology on $X_0$.

If $f{:}\,X \longrightarrow Y$ is a morphism in $(proAb, Ab)$, then the functor $L$ is defined by $Lf = f_0{:}\,X_0 \longrightarrow Y_0$. We must check that $f_0$ is continuous. Assume that $f{:}\,X \longrightarrow Y$ is given by a map $\varphi{:}\,\Lambda_Y \longrightarrow \Lambda_X$ $(\varphi 0 = 0)$ and homomorphisms $f_\mu{:}\,X_{\varphi(\mu)} \longrightarrow Y_\mu$. If $\mathrm{Im}Y^\mu_0$ is a neighbourhood at $0 \in Y_0$, there is a $\lambda \in \Lambda_X$ such that the following diagram is commutative

$$\begin{tikzcd}
& X_{\varphi(\mu)} \arrow[r, "f_\mu"] & Y_\mu \arrow[dd, "Y^\mu_0"] \\
X_\lambda \arrow[ur] \arrow[dr] & & \\
& X_0 \arrow[r, "f_0"'] & Y_0
\end{tikzcd}$$

This implies that $f_0(\mathrm{Im}X^\lambda_0) \subset \mathrm{Im}Y^\mu_0$. Therefore $Lf{:}\,LX \longrightarrow LY$ is a continuous homomorphism.

To define a functor $N{:}\,STAb \longrightarrow (proAb, Ab)$, for a given object $B$ of $STAb$ consider the directed set $\Lambda = \{S \mid S$ is a subgroup of $B$ and $S$ is a nbh at $0\ \}$ which has a final element $S = B$. Now define $NB{:}\,\Lambda \longrightarrow Ab$ by $NB_S = S, \quad S \in \Lambda$. Notice that $NB_0 = B$.

7.2. Proposition. *Consider the functors L and N defined above, then*

*1) $L{:}\,(proAb, Ab) \longrightarrow STAb$ is left adjoint to $N{:}\,STAb \longrightarrow (proAb, Ab)$*

*2) The unit, $B \longrightarrow LNB$, induced by the pair of adjoint functors, is a natural equivalence. Then $STAb$ can be considered as a full subcategory of $(proAb, Ab)$.*

Proof. Let $X$ be an object of $(proAb, Ab)$ and $Y$ an object of $STAb$. If $f{:}\,LX \longrightarrow Y$ is a continuous homomorphism, for each structured neighbourhood $S$ of $Y$, there exists a structured neighbourhood $\mathrm{Im}X^{\varphi S}_0$ at 0 such that $f(\mathrm{Im}X^{\varphi S}_0) \subset S$; for $S = Y$ we take $\varphi S = 0$. Define $f^b{:}\,X \longrightarrow NY$ by $f^b = (\varphi, f^b_S)$ where $f^b_S{:}\,X_{\varphi(S)} \longrightarrow S$ is the composition $f^b_S = (f|\mathrm{Im}X^{\varphi S}_0)X^{\varphi S}_0$.

For a given $g{:}\,X \longrightarrow NY$, define $g^\#{:}\,LX \longrightarrow Y$ by $g^\# = g_0$. Now it is easy to check that $(f^b)^\# = f$ and $(g^\#)^b = g$. ∎

If $STAb/fc$ denotes the full subcategory of $STAb$ determined by first countable topological abelian groups, we also have:

7.3. Proposition. *The restriction functors*
$L\colon (towAb, Ab) \longrightarrow STAb/fc$ *and*
$N\colon STAb/fc \longrightarrow (towAb, Ab)$ *satisfy*
*1) $L$ is left adjoint to $N$*
*2) The unit $B \longrightarrow LNB$ is a natural equivalence and $STAb/fc$ can be considered as a full subcategory of $(towAb, Ab)$.*

We also consider the following functors $g\colon STAb \longrightarrow Ab$ that forgets the topology and the functor $t\colon Ab \longrightarrow STAb$ defined as follows: If $A$ is an abelian group, $tA$ is the abelian group $A$ together with the trivial topology. Notice that $t$ is also a functor of the form $t\colon Ab \longrightarrow STAb/fc$. We have the following properties:

7.4. Proposition. *The functors above satisfy*
*1) $g\colon STAb \longrightarrow Ab$ is left adjoint to $t\colon Ab \longrightarrow STAb$*
*2) $g\colon STAb/fc \longrightarrow Ab$ is left adjoint to $t\colon Ab \longrightarrow STAb/fc$.*

Next we prove that the category $STAb/fc$ does not have countable sums. To do this, we take $X = Lc\mathbf{Z}_a$ that is given by the free abelian group

$$X = X(0) = f_a\{x_0, x_1, x_2, \ldots\}$$

and the local neighbourhood base at 0 given by

$$X(n) = f_a\{x_n, x_{n+1}, x_{n+2}, \ldots\}$$

In the category $STAb$, we can consider the countable sum $S = \bigoplus_{i=0}^{\infty} X_i$ together the topology given by the following local base. For each sequence $\mathbf{n} = (n_0, n_1, n_2, \ldots) \in$ $\in \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \cdots$ we consider

$$S(\mathbf{n}) = X(n_0) \oplus X(n_1) \oplus X(n_2) \oplus \cdots$$

It is not difficult to check that $S$ is the countable sum in the category $STAb$.

7.5. Lemma. $S = \bigoplus_{i=0}^{\infty} X_i$ *is a non first countable topological abelian group.*

Proof. Assume that we have a countable neighbourhood base at 0. This implies the existence of a sequence $\ldots \mathbf{m}^2 > \mathbf{m}^1 > \mathbf{m}^0$ in $\mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \cdots$, where
$\mathbf{m}^0 = (m_0^0, m_1^0, m_2^0, \ldots)$
$\mathbf{m}^1 = (m_0^1, m_1^1, m_2^1, \ldots)$
$\mathbf{m}^2 = (m_0^2, m_1^2, m_2^2, \ldots)$
$\ldots$
such that $S(\mathbf{m}^0) \supset S(\mathbf{m}^1) \supset S(\mathbf{m}^2) \supset \cdots$ is a countable neighbourhood base at 0, where
$S(\mathbf{m}^0) = X(m_0^0) \oplus X(m_1^0) \oplus X(m_2^0) \oplus \cdots$

$S(\mathbf{m}^1) = X(m_0^1) \oplus X(m_1^1) \oplus X(m_2^1) \oplus \cdots$
$S(\mathbf{m}^2) = X(m_0^2) \oplus X(m_1^2) \oplus X(m_2^2) \oplus \cdots$
$\ldots$

Now consider $\mathbf{m} = (m_0, m_1, m_2, \ldots)$ such that
$m_0 > m_0^0$
$m_1 > m_1^1 = max\{m_1^0, m_1^1\}$
$m_2 > m_2^2 = max\{m_2^0, m_2^1, m_2^2\}$
$\ldots$

Then we have that

$S(\mathbf{m}) = X(m_0) \oplus X(m_1) \oplus X(m_2) \oplus \cdots$

is a neighbourhood at 0 that does not contain $S(\mathbf{m}^i)$ for $i \geq 0$. This contradiction comes from the assumption that $S$ was first countable. Therefore $S$ is non first countable. ∎

7.6. Corollary. *The full subcategory $STAb/fc$ of $STAb$ is not closed under countable sums.*

7.7. Lemma. *The category $STAb/fc$ does not have the countable sum $\sum_{i=0}^{\infty} X$, where $X = Lc\mathbf{Z}_a$.*

Proof. Suppose that we have a first countable topological abelian group $\sum_{i=0}^{\infty} X_i$, where $X_i = X$ for $i \geq 0$. Let $in_i\colon X_i \longrightarrow \sum_{i=0}^{\infty} X_i$ be the canonical "inclusions". Now since $g\colon STAb/fc \longrightarrow Ab$ is a left adjoint by Proposition 7.3, it follows that $g$ preserves sums. Therefore there exists an isomorphism $\theta\colon g(\sum_{i=0}^{\infty} X_i) \longrightarrow \bigoplus_{i=0}^{\infty} gX_i$ such that for each $i \geq 0$, the following diagram is commutative

$$\begin{tikzcd} g(\sum_{i=0}^{\infty} X_i) \arrow[rr, "\theta"] & & \bigoplus_{i=0}^{\infty} gX_i \\ & gX_i \arrow[ul, "gin_i"] \arrow[ur, "in_{gX_i}"'] & \end{tikzcd}$$

where $gX_i = X$ for $i \geq 0$. The isomorphism $\theta$ induces a topology $\tau_f$ on $\bigoplus_{i=0}^{\infty} X_i$ such that $in_{X_i}\colon X_i \longrightarrow \bigoplus_{i=0}^{\infty} X_i$ is continuous for each $i \geq 0$. Then $(\bigoplus_{i=0}^{\infty} X_i, \tau_f)$ together with the "inclusions" is the sum in the category $STAb/fc$.

Let $(\bigoplus_{i=0}^{\infty} X_i, \tau_n)$ be the sum in the category $STAb$. Since $in_i\colon X_i \longrightarrow \bigoplus_{i=0}^{\infty} X_i$ is continuous for the topology $\tau_f$, it follows that $id\colon(\bigoplus_{i=0}^{\infty} X_i, \tau_n) \rightarrow (\bigoplus_{i=0}^{\infty} X_i, \tau_f)$ is continuous. Therefore $\tau_n$ is finer than $\tau_f$.

For each $\mathbf{n} = (n_0, n_1, n_2, \ldots)$, consider the abelian group $\overset{\infty}{\underset{i=0}{\oplus}} X_i/X(n_i)$ provided with the discrete topology, $dis$, which is first countable. As $in_i\colon X_i \longrightarrow \overset{\infty}{\underset{i=0}{\oplus}} X_i/X(n_i)$ is continuous for $i \geq 0$, the natural projection $p\colon (\overset{\infty}{\underset{i=0}{\oplus}} X_i, \tau_f) \to (\overset{\infty}{\underset{i=0}{\oplus}} X_i/X(n_i), dis)$ is continuous. Then $p^{-1}(0) = \overset{\infty}{\underset{i=0}{\oplus}} X(n_i)$ is an open neighbourhood of 0 for $\tau_f$. This implies that $\tau_f$ is finer than $\tau_n$. As $\tau_f = \tau_n$, we have $\tau_n$ is a first countable topology. This fact contradicts Lemma 7.5. ∎

7.8. Corollary. *$STAb/fc$ does not have countable sums.*

7.9. Corollary. *The category $(towAb, Ab)$ does not have the countable sum $\sum_{i=0}^{\infty} c\mathbf{Z}$.*

7.10. Corollary. *The category $(towAb, Ab)$ does not have countable sums.*

7.11. Corollary. *The category $(towAb, Ab)$ is not equivalent to a category of modules.*

Proof. Assume that the object $\sum_{i=0}^{\infty} c\mathbf{Z}$ exists. Since $L\colon (towAb, Ab) \longrightarrow STAb/fc$ is a left adjoint functor, it follows that $L$ preserves sums. Then $L(\sum_{i=0}^{\infty} c\mathbf{Z}) \cong \sum_{i=0}^{\infty} Lc\mathbf{Z}$. This contradicts Lemma 7.7. ∎

7.12. Corollary. *The category $towAb$ does not have the countable sum $\sum_{i=0}^{\infty} ({}^{\infty}c\mathbf{Z})$.*

7.13. Corollary. *The category $towAb$ does not have countable sums.*

7.14. Corollary. *The category $towAb$ is not equivalent to a category of modules.*

Proof. If there is a countable sum $\sum_{i=0}^{\infty} ({}^{\infty}c\mathbf{Z})$ in $towAb$, $\sum_{i=0}^{\infty} ({}^{\infty}c\mathbf{Z}) \longrightarrow \sum_{i=0}^{\infty} c\mathbf{Z}_0$ would be isomorphic to a countable sum $\sum_{i=0}^{\infty} c\mathbf{Z}$ in $(towAb, Ab)$. This is not possible by Corollary 7.9. ∎

Next we use topological abelian groups to prove that the extended functor $\mathcal{P}\colon proAb \longrightarrow Ab_{\mathcal{P}c\mathbf{Z_a}}$ is not faithful. This implies that $c\mathbf{Z}_a$ is not a generator of all the category $proAb$.

Let $S$ be an infinite set and let $f_aS$ be the free abelian group generated by $S$. Consider the topology defined on $f_aS$ by the family of subgroups of the form $f_aT$ where $T \subset S$ and $S - T$ is a countable set. We are going to see that if a sequence $y_k$ in $f_aT$ converges to zero, then there exists $k_0$ such that $y_k = 0$ for every $k \geq 0$. Assume that there is a subsequence $x_i = y_{k_i}$ with $x_i \neq 0$ for every i. Since $y_k \to 0$, it follows that $x_i \to 0$. Each $x_i$ can be written as a linear combination of finitely many elements of $S$. Therefore the sequence $x_i$ determines a countable set $S'$ of generators such that $x_i \notin f_a(S - S')$ for every $i \geq 0$. However this contradicts the fact that $x_i \to 0$.

Using the functor $N\colon STAb \longrightarrow (proAb, Ab)$ we have the global proobject $Nf_aS$ that also defines a proobject denoted in the same way in $proAb$.

7.15. Proposition. *$proAb(c\mathbf{Z}_a, Nf_aS) \cong 0$.*

PROOF. The hom-set $proAb(c\mathbf{Z}_a, Nf_aS)$ is a quotient of $(proAb, Ab)(c\mathbf{Z}_a, Nf_aS)$. By Proposition 7.2, $(proAb, Ab)(c\mathbf{Z}_a, Nf_aS) \cong STAb(Lc\mathbf{Z}_a, f_aS)$. Notice that $STAb(Lc\mathbf{Z}_a, f_aS)$ is the set of sequences in $f_aS$ converging to zero. Two converging sequences define the same morphism in $proAb(c\mathbf{Z}_a, Nf_aS)$ if and only if they have the same germ as the zero sequence. Therefore it follows that $proAb(c\mathbf{Z}_a, Nf_aS) \cong 0$. ∎

7.16. COROLLARY. *The functor* $\mathcal{P}: proAb \longrightarrow Ab_{\mathcal{P}c\mathbf{Z}_a}$ *is not faithful.*

PROOF. Since the bonding morphisms of $Nf_aS$ are non trivial, we have that $Nf_aS$ is not isomorphic to the zero object. This implies that the identity id of $Nf_aS$ is not equal to the zero map $0: Nf_aS \to Nf_aS$. By Proposition 7.15 we have that $\mathcal{P}(id) = \mathcal{P}(0) = 0$. ∎

7.17. REMARK. Grossman's result that $\mathcal{P}$ reflects isomorphisms does not work for the extended functor $\mathcal{P}: proAb \longrightarrow Ab_{\mathcal{P}c\mathbf{Z}_a}$.

7.18. COROLLARY. *The object* $c\mathbf{Z}$ *does not generate the whole of the category proAb.*

PROOF. It is easy to prove that if $c\mathbf{Z}$ were a generator for *proAb*, then $\mathcal{P}: proAb \longrightarrow Ab_{\mathcal{P}c\mathbf{Z}_a}$ would be a faithful functor. ∎

7.19. REMARK. At present, the author [He.2] is writing a paper that contains some topological applications of the embeddings given in this paper. It considers an extension of the $\mathcal{P}$ functor to categories whose objects are towers of simplicial sets or towers of simplicial groups. One of the main results of the new paper is the construction of a simplicial set ho$\mathcal{P}X$ associated with a tower of simplicial sets $X$. This space is constructed by considering a right-derived functor $ho\mathcal{P}$ of the version of Brown's $\mathcal{P}$ functor defined for the category of towers of simplicial sets. Recall that the homotopy limit functor, holim, can be defined as the right-derived functor of the lim functor. The simplicial set holim$X$ is a simplicial subset of ho$\mathcal{P}X$. It is well known that the Hurewicz homotopy groups of holim$X$ are the strong (or Steenrod) homotopy groups of the tower $X$, we obtain that the Hurewicz homotopy groups of the larger space ho$\mathcal{P}X$ are the Brown-Grossman homotopy groups of the tower $X$.

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