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CLOSEDNESS PROPERTIES OF INTERNAL RELATIONS I: A UNIFIED APPROACH TO MAL'TSEV, UNITAL AND SUBTRACTIVE CATEGORIES

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ABSTRACT. We study *closedness properties* of internal relations in finitely complete categories, which leads to developing a unified approach to: Mal'tsev categories, in the sense of A. Carboni, J. Lambek and M. C. Pedicchio, that generalize Mal'tsev varieties of universal algebras; unital categories, in the sense of D. Bourn, that generalize pointed Jónsson-Tarski varieties; and subtractive categories, introduced by the author, that generalize pointed subtractive varieties in the sense of A. Ursini.

Introduction

The notion of a subtractive category, introduced in [12], extends the notion of a subtractive variety of universal algebras, due to A. Ursini [22], to abstract pointed categories. Subtractive categories are closely related to Mal'tsev categories in the sense of A. Carboni, J. Lambek, and M. C. Pedicchio [7] (see also [8] and [6]), which generalize Mal'tsev varieties [18], and to unital categories in the sense of D. Bourn [4] (see also [5]), which generalize pointed Jónsson-Tarski varieties [16]. In the present paper, which is based on the author's M.Sc. Thesis [11] (see also [13]), we develop a unified approach to these three classes of categories. In particular, we show that *the procedure of forming these classes of categories from the corresponding classes of varieties is the same in all the three cases.*

The main tool that we use is a new notion of an M-closed relation, where M is an extended matrix of terms of an algebraic theory. The main observation is that each one of the above classes of categories can be obtained as the class of *finitely complete categories (or pointed categories) with M-closed relations*, where, in each case, the matrix M is naturally obtained from the term condition that determines the corresponding class of varieties.

We also give several characterizations of categories with M-closed relations, and then apply them, in particular, to the case of Mal'tsev categories. We show that several known characterizations of Mal'tsev categories can be obtained in this way.

The paper consists of six sections. The first section is devoted to the notion of M-

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closed relation. It contains several technical propositions whose special instances are often encountered in the literature on Mal'tsev categories. In the second section we define (using the Yoneda embedding) M-closedness of internal relations in a category. We also introduce a notion of an M-closed functor, which allows a unified treatment of *M-closed interpretations* introduced in the third section, and *M-closed $\mathcal{T}$-enrichments* introduced in the fourth section. The fifth section contains the definition and characterizations of a category with M-closed relations. In the sixth section we prove that Mal'tsev, unital, strongly unital and subtractive categories are the same as categories (pointed categories) with M-closed relations, where in each case M is obtained from the syntactical condition defining the corresponding varieties.

In the sequel [14] of this paper, we will extend Bourn's characterization theorem for Mal'tsev categories, involving *fibration of points* [4], to categories with M-closed relations, where M is an arbitrary matrix of terms of the algebraic theory of sets. The general characterization theorem will also include, as its another special case, the characterization of Mal'tsev categories obtained in [12]. Other future papers from this series will contain the following topics:

- How to express (pointed) protomodularity in the sense of D. Bourn [3] using the notion of an M-closed relation (this will involve matrices which can be obtained from a characterization of "BIT speciale" varieties in the sense of A. Ursini [21], which in the pointed case are the same as protomodular varieties, given in Ursini's paper [22]).

- For a general term matrix M, how are the following two conditions on a category $\mathcal{C}$ related to each other: (a) $\mathcal{C}$ is enriched in the variety of commutative M-algebras, (b) internal relations both in $\mathcal{C}$ and in $\mathcal{C}^{\mathrm{op}}$ are M-closed. For instance, in the case when M is the matrix which determines the class of unital categories, these two conditions are equivalent to each other (for a category $\mathcal{C}$ having finite limits and binary sums), and they define half-additive categories in the sense of P. Freyd and A. Scedrov[10]. These two conditions are equivalent also when M is the matrix which determines the class of Mal'tsev categories; then they define naturally Mal'tsev categories in the sense of P. T. Johnstone [15].

- How to translate an arbitrary linear Mal'tsev condition in the sense of J. W. Snow [20] to a categorical condition (this will involve closedness properties with respect to extended matrices of variables with distinguished entries).

Convention. Throughout the paper by a category we will always mean a category having finite limits.

1. M-closed relations

Let $A_1, ..., A_n$ be arbitrary sets. We will consider extended matrices

$$M = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right),$$

where $a_{i1}, ..., a_{im}, b_i \in A_i$ for each $i \in \{1, ..., n\}$; we then say that M is an $n \times (m+1)$ extended matrix with columns from $A_1 \times ... \times A_n$. Here we assume $n \geqslant 1$ and $m \geqslant 0$. If $m = 0$ then M becomes

$$\left(\begin{array}{c|c} & b_1 \\ & \vdots \\ & b_n \end{array}\right);$$

in this case we say that M is degenerate. The columns of a's will be called left columns of M, and the column of b's will be called the right column of M. Note that M always has the right column, and it has a left column if and only if it is *nondegenerate*.

1.1. Definition. *A relation $R \subseteq A_1 \times ... \times A_n$ is said to be compatible with an extended matrix*

$$M = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right)$$

with columns from $A_1 \times ... \times A_n$, if whenever R contains every left column of M, it also contains the right column of M, i.e. if

$$\left\{ \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, ..., \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} \right\} \subseteq R \implies \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in R.$$

If M is degenerate, then R is compatible with M if and only if the right column of M is an element of R.

Consider maps

$$f_i : A_i \to A'_i, \ \ i \in \{1, ..., n\}.$$

M gives rise to an extended matrix

$$(f_1, ..., f_n)_! M = \left(\begin{array}{ccc|c} f_1(a_{11}) & \cdots & f_1(a_{1m}) & f_1(b_1) \\ \vdots & & \vdots & \vdots \\ f_n(a_{n1}) & \cdots & f_n(a_{nm}) & f_n(b_n) \end{array}\right)$$

with columns from $A'_1 \times ... \times A'_n$. At the same time, a relation $R \subseteq A'_1 \times ... \times A'_n$ gives rise, via the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & R \arrow[d] \\ A_1 \times ... \times A_n \arrow[r, "f_1 \times ... \times f_n"'] & A'_1 \times ... \times A'_n \end{tikzcd}$$

to a relation $(f_1, ..., f_n)^* R = P \subseteq A_1 \times ... \times A_n$.

1.2. Lemma. *The relation $(f_1, ..., f_n)^* R$ is compatible with the extended matrix M if and only if the relation R is compatible with the extended matrix $(f_1, ..., f_n)_! M$.*

Let $\mathcal{T}$ be an algebraic theory and let $\mathcal{X}$ denote the alphabet of $\mathcal{T}$. We will not distinguish between terms of $\mathcal{T}$ and the corresponding elements of the free $\mathcal{T}$-algebra $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$ over $\mathcal{X}$. Let M be an extended matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array}\right)$$

of terms t_{ij}, u_i of $\mathcal{T}$, and let $A_1, ..., A_n$ be $\mathcal{T}$-algebras.

1.3. Definition. *An $n \times (m+1)$ extended matrix M' with columns from $A_1 \times ... \times A_n$ is said to be a row-wise interpretation of M if there exist $\mathcal{T}$-algebra homomorphisms*

$$f_1 : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A_1, ..., f_n : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A_n$$

such that $M' = (f_1, ..., f_n)_! M$. Suppose $A_1 = ... = A_n = A$, then M' is said to be a regular interpretation of M if we could take $f_1 = ... = f_n$ above, i.e. if there exists a $\mathcal{T}$-algebra homomorphism $f : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A$ such that $M' = (f, ..., f)_! M$.

Since M has only a finite number of entries, there exists a sequence $x_1, ..., x_k$ of some k number of distinct variables such that each term from M depends only on those variables that are members of this sequence; the choice of each such sequence $x_1, ..., x_k$ allows to regard each term in M as a k-ary term, and then we can write:

- M' is a row-wise interpretation of M if

$$M' = \left(\begin{array}{ccc|c} t_{11}(c_{11}, ..., c_{1k}) & \cdots & t_{1m}(c_{11}, ..., c_{1k}) & u_1(c_{11}, ..., c_{1k}) \\ \vdots & & \vdots & \vdots \\ t_{n1}(c_{n1}, ..., c_{nk}) & \cdots & t_{nm}(c_{n1}, ..., c_{nk}) & u_n(c_{n1}, ..., c_{nk}) \end{array}\right)$$

  for some $c_{11}, ..., c_{1k} \in A_1, ..., c_{n1}, ..., c_{nk} \in A_n$;

- M' is a regular interpretation of M if

$$M' = \left(\begin{array}{ccc|c} t_{11}(c_1,...,c_k) & \cdots & t_{1m}(c_1,...,c_k) & u_1(c_1,...,c_k) \\ \vdots & & \vdots & \vdots \\ t_{n1}(c_1,...,c_k) & \cdots & t_{nm}(c_1,...,c_k) & u_n(c_1,...,c_k) \end{array}\right)$$

for some $c_1, ..., c_k \in A$.

1.4. Definition. *An n-ary relation R on a $\mathcal{T}$-algebra A is said to be closed with respect to M (or M-closed) if R is compatible with every extended matrix of elements of A that is a regular interpretation of M. An n-ary relation R between $\mathcal{T}$-algebras $A_1, ..., A_n$ is said to be strictly closed with respect to M (or strictly M-closed) if R is compatible with every extended matrix with columns from $A_1 \times ... \times A_n$ that is a row-wise interpretation of M.*

Note that a strictly M-closed relation $R \subseteq A^n$ is always M-closed.

1.5. Examples. Consider the case when $\mathcal{T}$ is the algebraic theory corresponding to the variety of sets, $\mathcal{T} = \mathrm{Th}[sets]$. Then $\mathrm{Fr}_{\mathcal{T}}\mathcal{X} = \mathcal{X}$ and so each entry of M is a variable. M' is a regular interpretation of M if and only if whenever two entries in M coincide, the corresponding entries in M' also coincide. M' is a row-wise interpretation of M if and only if whenever in each row of M two entries coincide, the corresponding entries of the corresponding row of M' also coincide.

Reflexivity, symmetry and transitivity of a binary relation are examples of closedness with respect to an extended matrix of variables — see Table 1, where we also exhibit the strict versions of these three closedness properties.

A binary relation $R \subseteq A \times B$ between sets A, B is said to be *difunctional* if it is strictly closed with respect to

$$\left(\begin{array}{ccc|c} x & y & y & x \\ u & u & v & v \end{array}\right). \tag{1}$$

That is, R is difunctional if it satisfies

$$a_1Rb_1 \wedge a_2Rb_1 \wedge a_2Rb_2 \quad \Rightarrow \quad a_1Rb_2$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. For $A = B$ difunctionality of R can be also defined as closedness (without "strict") with respect to (1).

Let $\mathcal{T}$ be again an arbitrary algebraic theory. Suppose all entries of M are variables, i.e. they are elements of the alphabet $\mathcal{X}$ of $\mathcal{T}$. Then the corresponding closedness properties of a relation $R \subseteq A_1 \times ... \times A_n$ do not depend on the $\mathcal{T}$-algebra structures of $A_1, ..., A_n$, and they are the same as for M regarded as a matrix of terms in $\mathrm{Th}[sets]$ (and the A's regarded as sets). More generally, consider an interpretation ι of $\mathcal{T}$ in an algebraic theory $\mathcal{T}'$. For a term w in $\mathcal{T}$ let us write w^ι for the corresponding term of $\mathcal{T}'$. An extended matrix M of terms in $\mathcal{T}$ gives rise to an extended matrix

$$M^\iota = ((-)^\iota, ..., (-)^\iota)_! M = \left(\begin{array}{ccc|c} t^\iota_{11} & \cdots & t^\iota_{1m} & u^\iota_1 \\ \vdots & & \vdots & \vdots \\ t^\iota_{n1} & \cdots & t^\iota_{nm} & u^\iota_n \end{array}\right)$$

$M =$	$R \subseteq A \times A$ is M-closed iff	$R \subseteq A \times B$ is strictly M-closed iff
$\left(\begin{array}{c\|c} & x \\ & x \end{array}\right)$	R is reflexive	$R = A \times B$
$\left(\begin{array}{c\|c} x & y \\ y & x \end{array}\right)$	R is symmetric	$R = A \times B$ or $R = \varnothing$
$\left(\begin{array}{cc\|c} x & y & x \\ y & z & z \end{array}\right)$	R is transitive	$R = A' \times B'$ for some $A' \subseteq A$ and $B' \subseteq B$

Table 1: Reflexivity, symmetry and transitivity matrices

of terms in $\mathcal{T}'$. Let ι^* denote the forgetful functor $\iota^* : \mathbf{Alg}_{\mathcal{T}'} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ corresponding to the interpretation ι. A relation R on a $\mathcal{T}'$-algebra A is M^ι-closed if and only if R is M-closed as a relation on the $\mathcal{T}$-algebra $\iota^*(A)$. Similarly, a relation R between $\mathcal{T}'$-algebras $A_1, ..., A_n$ is strictly M^ι-closed if and only if R is strictly M-closed as a relation between $\mathcal{T}$-algebras $\iota^*(A_1), ..., \iota^*(A_1)$.

1.6. Examples. Suppose $\mathcal{T}$ is the algebraic theory corresponding to the variety of pointed sets, $\mathcal{T} = \mathrm{Th}[\textit{pointed sets}]$. Let 0 denote the unique nullary term of this theory. Each entry of M is now either a variable or 0. An extended matrix M' of elements of a pointed set A is a regular interpretation of M if and only if whenever two entries in M coincide, the corresponding entries in M' also coincide, and an entry in M is 0 implies that the corresponding entry in M' is the base point of A. An extended matrix M' whose each i-th row consists of elements of a pointed set A_i is a row-wise interpretation of M if and only if whenever in each row of M two entries coincide, the corresponding entries of the corresponding row of M' also coincide, and an entry of some i-th row of M is 0 implies that the corresponding entry of the i-th row of M' is the base point of A_i.

A binary relation R on a pointed set A is 0-*transitive* in the sense of P. Agliano and A. Ursini [1] if R is closed with respect to the matrix

$$\left(\begin{array}{cc|c} 0 & y & 0 \\ y & z & z \end{array}\right).$$

A binary relation R on a pointed set A is 0-*symmetric* in the sense of P. Agliano and A. Ursini [1] if R and its inverse relation R^{op} are closed with respect to the matrix

$$\left(\begin{array}{c|c} x & 0 \\ 0 & x \end{array}\right).$$

For an arbitrary $\mathcal{T}$ and M we have:

1.7. Lemma.

(a) An n-ary relation R on a $\mathcal{T}$-algebra A is M-closed if and only if for any $\mathcal{T}$-algebra homomorphism $f : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A$, the n-ary relation $(f_1, ..., f_n)^* R$ on $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$ is compatible with M.

(b) An n-ary relation R between $\mathcal{T}$-algebras $A_1, ..., A_n$ is strictly M-closed if and only if for any $\mathcal{T}$-algebra homomorphisms $f_1 : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A_1, ..., f_n : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A_n$, the n-ary relation $(f_1, ..., f_n)^* R$ on $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$ is compatible with M.

The following lemma can be easily derived from the lemma above, using the general fact that in a diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

if the two small rectangles are pullbacks, then the large rectangle is also a pullback.

1.8. Lemma. *For each $i \in \{1, ..., n\}$, let A_i, A_i' be $\mathcal{T}$-algebras and let $f_i : A_i \longrightarrow A_i'$ be a $\mathcal{T}$-algebra homomorphism. Let R be a relation between $A_1', ..., A_n'$.*

(a) If R is strictly M-closed then $(f_1, ..., f_n)^* R$ is strictly M-closed.

(b) Suppose $A_1 = ... = A_n$, $A_1' = ... = A_n'$, $f_1 = ... = f_n = f$. Then, R is M-closed implies $(f, ..., f)^* R$ is M-closed.

The following proposition shows how to express strict M-closedness via M-closedness:

1.9. Proposition. *Let R be a relation between $\mathcal{T}$-algebras $A_1, ..., A_n$ and let*

$$S = (\pi_1, ..., \pi_n)^* R,$$

where π_i denotes the i-th projection $A_1 \times ... \times A_n \to A_i$, i.e. S is the n-ary relation on $A_1 \times ... \times A_n$ obtained via the pullback

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & R \arrow[d] \\
(A_1 \times ... \times A_n)^n \arrow[r, "\pi_1 \times ... \times \pi_n"'] & A_1 \times ... \times A_n
\end{tikzcd}$$

The following conditions are equivalent:

(a) R is strictly M-closed.

(b) S is strictly M-closed.

(c) S is M-closed.

PROOF. (a) $\Rightarrow$ (b) follows from Lemma 1.8, and (b) $\Rightarrow$ (c) is trivial. (c) $\Rightarrow$ (a) follows from Lemma 1.7 and the observation that each pullback

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & R \arrow[d] \\ (\mathrm{Fr}_{\mathcal{T}}\mathcal{X})^n \arrow[r, "f_1\times\ldots\times f_n"'] & A_1\times\ldots\times A_n \end{tikzcd}$$

can be decomposed into the following two pullbacks:

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & R \arrow[d] \\ (\mathrm{Fr}_{\mathcal{T}}\mathcal{X})^n \arrow[r, "(f_1,\ldots,f_n)^n"'] & (A_1\times\ldots\times A_n)^n \arrow[r, "\pi_1\times\ldots\times\pi_n"'] & A_1\times\ldots\times A_n \end{tikzcd}$$

■

The notion of reflexivity of a binary relation can be naturally extended to relations of an arbitrary arity. An n-ary relation on a set A is said to be reflexive if it is closed with respect to the degenerate matrix

$$\left(\begin{array}{c|c} & x \\ & \vdots \\ & x \end{array}\right)$$

whose all entries are the same variable x.

It is easy to see that for any relation $R \subseteq A_1\times\ldots\times A_n$, the n-ary relation $(r_1,\ldots,r_n)^*R$ on R, induced by the projections $r_i : R \longrightarrow A_i$, is a reflexive relation.

1.10. PROPOSITION. *Suppose*

(*) *there exist m-ary terms $p_1,\ldots,p_k$ in $\mathcal{T}$ such that*

$$t_{ij}(p_1(t_{i1},\ldots,t_{im}),\ldots,p_k(t_{i1},\ldots,t_{im})) = t_{ij},$$
$$u_i(p_1(t_{i1},\ldots,t_{im}),\ldots,p_k(t_{i1},\ldots,t_{im})) = u_i$$

for each $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$.

Then, for any n-ary homomorphic relation R between $\mathcal{T}$-algebras, the following conditions are equivalent:

(a) R is strictly M-closed.

(b) $(r_1,\ldots,r_n)^*R$ is strictly M-closed.

(c) $(r_1,\ldots,r_n)^*R$ is M-closed.

Proof. (a) $\Rightarrow$ (b) follows from Lemma 1.8, and (b) $\Rightarrow$ (c) is trivial. For (c) $\Rightarrow$ (a) it suffices to show that

(*′) for any row-wise interpretation M' of M with columns from $A_1 \times ... \times A_n$, such that all of the left columns of M' are elements of R, there exists a regular interpretation M'' of M with columns from R^n such that $M' = (r_1, ..., r_n)_! M''$.

The condition (*′) states that

(*″) for any

$$c_{11}, ..., c_{1k} \in A_1, ..., c_{n1}, ..., c_{nk} \in A_n$$

such that

$$\begin{pmatrix} t_{1j}(c_{11}, ..., c_{1k}) \\ \vdots \\ t_{nj}(c_{n1}, ..., c_{nk}) \end{pmatrix} \in R$$

for all $j \in \{1, ..., m\}$, there exist $d_1, ..., d_k \in R$ such that

$$r_i(t_{ij}(d_1, ..., d_k)) = t_{ij}(c_{i1}, ..., c_{ik}) \text{ and } r_i(u_i(d_1, ..., d_k)) = u_i(c_{i1}, ..., c_{ik})$$

for each $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$.

For each $l \in \{1, ..., k\}$ we take

$$\begin{aligned} d_l &= p_l(\begin{pmatrix} t_{11}(c_{11}, ..., c_{1k}) \\ \vdots \\ t_{n1}(c_{n1}, ..., c_{nk}) \end{pmatrix}, ..., \begin{pmatrix} t_{1m}(c_{11}, ..., c_{1k}) \\ \vdots \\ t_{nm}(c_{n1}, ..., c_{nk}) \end{pmatrix}) \\ &= \begin{pmatrix} p_l(t_{11}(c_{11}, ..., c_{1k}), ..., t_{1m}(c_{11}, ..., c_{1k})) \\ \vdots \\ p_l(t_{n1}(c_{n1}, ..., c_{nk}), ..., t_{nm}(c_{n1}, ..., c_{nk})) \end{pmatrix} \end{aligned}$$

with p_l the same as in (*). Since R is a homomorphic relation, $d_1, ..., d_k \in R$. Moreover,

$$\begin{aligned} r_i(t_{ij}(d_1, ..., d_k)) &= t_{ij}(\, p_1(t_{i1}(c_{i1}, ..., c_{ik}), ..., t_{im}(c_{i1}, ..., c_{ik})), ... \\ &\quad ..., p_k(t_{i1}(c_{i1}, ..., c_{ik}), ..., t_{im}(c_{i1}, ..., c_{ik}))\,) \\ &= t_{ij}(c_{i1}, ..., c_{ik}), \\ r_i(u_i(d_1, ..., d_k)) &= u_i(\, p_1(t_{i1}(c_{i1}, ..., c_{ik}), ..., t_{im}(c_{i1}, ..., c_{ik})), ... \\ &\quad ..., p_k(t_{i1}(c_{i1}, ..., c_{ik}), ..., t_{im}(c_{i1}, ..., c_{ik}))\,) \\ &= u_i(c_{i1}, ..., c_{ik}), \end{aligned}$$

as desired. ∎

It is easy to show that for $\mathcal{T} = \mathrm{Th}[sets]$, the condition (*) is equivalent to the following condition:

(**) For each variable x there exists a left column

$$\begin{pmatrix} t_{1j} \\ \vdots \\ t_{nj} \end{pmatrix}$$

in M such that for each $i \in \{1, ..., n\}$, if the i-th row of M contains x then $t_{ij} = x$.

It is easy to show also that for $\mathcal{T} = \mathrm{Th}[pointed\ sets]$ the condition (*) is satisfied if and only if either all entries of M are 0's, or (**) is satisfied.

Note that the condition (**) is satisfied for a difunctionality matrix, but not for reflexivity, symmetry or transitivity matrices.

1.11. Remark. If (*) is not satisfied then the implications 1.10(c) ⇒ 1.10(b), 1.10(b) ⇒ 1.10(a) and 1.10(c) ⇒ 1.10(a) need not be satisfied either. Indeed, take $\mathcal{T} = \mathrm{Th}[sets]$ and

$$M = \left(\begin{array}{c|c} & x \\ & x \end{array} \right).$$

Then

- the condition (**) is not satisfied;
- a relation $R \subseteq A \times B$ is strictly M-closed if and only if it is codiscrete, i.e. $R = A \times B$;
- the relation $(r_1, r_2)^* R$ on R is M-closed, for any relation $R \subseteq A \times B$, and $(r_1, r_2)^* R = R \times R$ if and only if R is strictly closed with respect to a transitivity matrix, i.e. $R = A' \times B'$ for some $A' \subseteq A$ and $B' \subseteq B$.

Thus,

- 1.10(a) is satisfied for all codiscrete binary relations,
- 1.10(b) is satisfied for all binary relations $R \subseteq A \times B$ such that $R = A' \times B'$ for some $A' \subseteq A$ and $B' \subseteq B$,
- 1.10(c) is satisfied for all binary relations.

2. Internal M-closedness and M-closed functors

Let $\mathcal{C}$ be a category and let A be an object in $\mathcal{C}$, equipped with an internal $\mathcal{T}$-algebra structure. The $\mathcal{T}$-algebra structure on A induces a $\mathcal{T}$-algebra structure on $\hom(X, A)$, for every object X in $\mathcal{C}$. An internal n-ary relation $R \longrightarrow A^n$ in $\mathcal{C}$ is said to be M-closed if for every object X, the induced relation $\hom(X, R)$ on the $\mathcal{T}$-algebra $\hom(X, A)$ is M-closed in the sense of Definition 1.4. Strict M-closedness of internal relations is defined analogously. For $\mathcal{C} = \mathbf{Set}$, internal M-closedness is the same as ordinary M-closedness (and the same is true for strict M-closedness).

The internal versions of Lemma 1.8 and Propositions 1.9 and 1.10 remain true.

We will now show that finite limit preserving functors always preserve M-closedness of internal relations, and those functors that in addition reflect isomorphisms also reflect M-closedness.

To each internal n-ary relation

$$r = (r_1, ..., r_n) : R \longrightarrow A^n$$

in $\mathcal{C}$, where A is an object of $\mathcal{C}$, equipped with an internal $\mathcal{T}$-algebra structure, we associate an internal k-ary relation (recall that k denotes the arity of the terms in M)

$$r^M = (r_1^M, ..., r_k^M) : R^M \longrightarrow A^k$$

where R^M is the object obtained as the limit of the diagram

$$\begin{tikzcd}
A^k \arrow[r, equal] \arrow[d, "\begin{pmatrix} t_{11} \\ \vdots \\ t_{n1} \end{pmatrix}"] & A^k \arrow[r, equal] \arrow[d, "\begin{pmatrix} t_{12} \\ \vdots \\ t_{n2} \end{pmatrix}"] & \cdots \arrow[r, equal] & A^k \arrow[d, "\begin{pmatrix} t_{1m} \\ \vdots \\ t_{nm} \end{pmatrix}"] \\
A^n & A^n & \cdots & A^n \\
R \arrow[u, "r"] & R \arrow[u, "r"] & \cdots & R \arrow[u, "r"]
\end{tikzcd}$$

(here t_{ij} denotes the operation $t_{ij} : A^k \longrightarrow A$ of the $\mathcal{T}$-algebra structure of A, corresponding to the term t_{ij}) and r^M is the limit projection $R^M \longrightarrow A^k$.

2.1. Lemma. *The following conditions are equivalent:*

- *(a) The relation R is M-closed.*
- *(b) The morphism $(u_1(r_1^M, ..., r_k^M), ..., u_n(r_1^M, ..., r_k^M)) : R^M \longrightarrow A^n$ factors through r.*

Let $F : \mathcal{C} \to \mathcal{D}$ be a finite limit preserving functor from $\mathcal{C}$ to a category $\mathcal{D}$ and let

$$r = (r_1, ..., r_n) : R \longrightarrow A_1 \times ... \times A_n$$

be an internal relation in $\mathcal{C}$, where $A_1, ..., A_n$ are objects of $\mathcal{C}$, each one equipped with an internal $\mathcal{T}$-algebra structure. R gives rise to an internal relation

$$(F(r_1), ..., F(r_n)) : F(R) \longrightarrow F(A_1) \times ... \times F(A_n)$$

in $\mathcal{D}$, while the internal $\mathcal{T}$-algebra structures on $A_1, ..., A_n$ give rise to internal $\mathcal{T}$-algebra structures on $F(A_1), ..., F(A_n)$.

2.2. Proposition.

(a) If the relation R is strictly M-closed then the relation $F(R)$ is also strictly M-closed. If F reflects isomorphisms, then R is strictly M-closed if and only if $F(R)$ is strictly M-closed.

(b) Suppose $A_1 = ... = A_n = A$. If R is M-closed then $F(R)$ is M-closed. If F reflects isomorphisms, then R is M-closed if and only if $F(R)$ is M-closed.

Proof. (b) follows from Lemma 2.1 and the fact that there is a canonical isomorphism $F(R^M) \simeq (F(R))^M$. (a) follows from (b) and (the internal version of) Proposition 1.9. ∎

Now let F be a finite limit preserving functor $F : \mathcal{C} \longrightarrow \mathbf{Alg}_\mathcal{T}\mathcal{D}$ from $\mathcal{C}$ to the category $\mathbf{Alg}_\mathcal{T}\mathcal{D}$ of internal $\mathcal{T}$-algebras in $\mathcal{D}$. Let U denote the forgetful functor $U : \mathbf{Alg}_\mathcal{T}\mathcal{D} \longrightarrow \mathcal{D}$.

2.3. Definition. *An internal relation $R \longrightarrow A^n$ in $\mathcal{C}$ is said to be (M, F)-closed if the corresponding internal relation $UF(R) \longrightarrow UF(A)^n$ in $\mathcal{D}$ is M-closed with respect to the internal $\mathcal{T}$-algebra structure of $F(A)$.*

An internal relation $R \longrightarrow A_1 \times ... \times A_n$ in $\mathcal{C}$ is said to be strictly (M, F)-closed if the corresponding internal relation $UF(R) \longrightarrow UF(A_1) \times ... \times UF(A_n)$ in $\mathcal{D}$ is strictly M-closed with respect to the internal $\mathcal{T}$-algebra structures of $F(A_1), ..., F(A_n)$.

The following theorem can be easily proved using the internal versions of Propositions 1.9 and 1.10.

2.4. Theorem. *The following conditions are equivalent:*

(a) Every relation $R \longrightarrow A_1 \times ... \times A_n$ in $\mathcal{C}$ is strictly (M, F)-closed.

(b) Every relation $R \longrightarrow A^n$ in $\mathcal{C}$ is (M, F)-closed.

If M satisfies the condition (*) from Section 1, then the conditions above are also equivalent to the following condition:

(c) Every reflexive relation $R \longrightarrow A^n$ in $\mathcal{C}$ is (M, F)-closed.

2.5. Definition. *The functor $F : \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}\mathcal{D}$ is said to be M-closed if the equivalent conditions (a),(b) in Theorem 2.4 are satisfied.*

3. M-closed interpretations

The extended term matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array} \right)$$

determines a system

$$\begin{cases} p(t_{11}, ..., t_{1m}) = u_1, \\ \vdots \\ p(t_{n1}, ..., t_{nm}) = u_n \end{cases} \tag{2}$$

of term equations, where p is the "unknown" term. Suppose $\mathcal{T}$ is the algebraic theory of the variety of (right) modules over a ring K. Then, each m-ary term p in $\mathcal{T}$ is of the form $p(x_1, ..., x_j) = x_1 \cdot p_1 + ... + x_m \cdot p_m$, for some uniquely determined $p_1, ..., p_m \in K$. If the entries of M are unary terms, all of which depend on the same variable, then we can regard M as an extended matrix of elements of K, and then the system of equations (2) becomes the system of linear equations corresponding to M:

$$\begin{cases} t_{11} \cdot p_1 + ... + t_{1m} \cdot p_m = u_1, \\ \vdots \\ t_{n1} \cdot p_1 + ... + t_{nm} \cdot p_m = u_n, \end{cases}$$

with $p_1, ..., p_n$ the "unknowns".

Let $\mathcal{T}$ be again an arbitrary algebraic theory.

We will say that the system (2) of term equations is solvable in $\mathcal{T}$, if there exists an m-ary term p in $\mathcal{T}$, for which these equations would be satisfied. Such p will be called a solution of (2) in $\mathcal{T}$.

3.1. Theorem. *The following conditions are equivalent:*

(a) Every n-ary homomorphic relation $R \subseteq A^n$ is M-closed, for any $\mathcal{T}$-algebra A (i.e. the identity functor $\mathbf{Alg}_{\mathcal{T}} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ is M-closed).

(b) The system of term equations (2), corresponding to M, is solvable in $\mathcal{T}$.

Proof. (a) is satisfied if and only if every n-ary homomorphic relation on $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$ is compatible with M (see Lemma 1.7), which is the case if and only if the smallest n-ary homomorphic relation on $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$ that contains all of the left columns of M, also contains the right column of M. This is equivalent to the existence of an m-ary term p in $\mathcal{T}$, for which

$$p(\begin{pmatrix} t_{11} \\ \vdots \\ t_{n1} \end{pmatrix}, ..., \begin{pmatrix} t_{1m} \\ \vdots \\ t_{nm} \end{pmatrix}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Rewriting this equality component-wise we obtain the equalities (2). ∎

Let ι be an interpretation of $\mathcal{T}$ in an algebraic theory $\mathcal{T}'$. We will say that ι is M-closed, if the system of term equations

$$\begin{cases} p(t^{\iota}_{11}, ..., t^{\iota}_{1m}) = u^{\iota}_1, \\ \vdots \\ p(t^{\iota}_{n1}, ..., t^{\iota}_{nm}) = u^{\iota}_n, \end{cases}$$

corresponding to the matrix M^{ι} is solvable in $\mathcal{T}'$.

3.2. Theorem. *The following conditions are equivalent:*

(a) The interpretation $\iota : \mathcal{T} \longrightarrow \mathcal{T}'$ is M-closed.

(b) The forgetful functor $\iota^* : \mathbf{Alg}_{\mathcal{T}'} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ is M-closed.

Proof. This follows easily from Theorem 3.1 (applied to the pair $\mathcal{T}', M^{\iota}$) and the observation that ι^* is M-closed if and only if the identity functor $\mathbf{Alg}_{\mathcal{T}'} \longrightarrow \mathbf{Alg}_{\mathcal{T}'}$ is M^{ι}-closed. ∎

Let $\mathbf{Var}_M$ denote the class of all varieties $\mathcal{V}$ of universal algebras for which there exists an M-closed interpretation $\iota : \mathcal{T} \longrightarrow \mathrm{Th}[\mathcal{V}]$. Many classes of varieties studied in universal algebra are of this form. In particular, for $\mathcal{T} = \mathrm{Th}[sets]$ and M equal to a difunctionality matrix

$$M = \left(\begin{array}{ccc|c} x & y & y & x \\ u & u & v & v \end{array}\right)$$

$\mathbf{Var}_M$ is precisely the class of Mal'tsev varieties. The system of term equations corresponding to this M is

$$\begin{cases} p(x, y, y) = x, \\ p(u, u, v) = v. \end{cases}$$

Its solution p is called a *Mal'tsev term*, and the equations above are called *Mal'tsev identities*. More generally, the equations in the system obtained from any matrix M', such that M and M' are each other's row-wise interpretations, are called Mal'tsev identities (note that such system of equations has the same set of solutions as the the system above); we will call such matrix M' a *Mal'tsev matrix*.

Although strict closedness of a relation with respect to a Mal'tsev matrix is the same as difunctionality, closedness with respect to a Mal'tsev matrix can be a much weaker property. For instance, any reflexive binary relation is closed with respect to the Mal'tsev matrix

$$\left(\begin{array}{ccc|c} x & y & y & x \\ y & y & x & x \end{array}\right),$$

however, *a reflexive relation is difunctional if and only if it is an equivalence relation.*

The theory of M-closed relations can be used to obtain various characterizations of Mal'tsev varieties. In particular, we obtain from Theorems 3.2 and 2.4 the following characterization theorem:

3.3. Corollary (J. Lambek [17]). *A variety $\mathcal{V}$ of universal algebras is a Mal'tsev variety (i.e. its theory contains a Mal'tsev term) if and only if the following equivalent conditions are satisfied:*

(a) Every homomorphic relation $R \subseteq A \times B$ in $\mathcal{V}$ is difunctional.

(b) Every homomorphic relation $R \subseteq A \times A$ in $\mathcal{V}$ is difunctional.

Consider the following two Mal'tsev matrices:

$$\left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array}\right), \quad \left(\begin{array}{ccc|c} x & y & y & x \\ y & y & v & v \end{array}\right).$$

A reflexive relation $R \subseteq A^2$ on a set A is closed with respect to the first matrix if and only if R is symmetric, and R is closed with respect to the second matrix if and only if R is transitive. Both of those matrices satisfy (**). From Theorem 2.4 we get:

3.4. Corollary (G. O. Findlay [9]). *A variety $\mathcal{V}$ of universal algebras is a Mal'tsev variety if and only if the following equivalent conditions are satisfied:*

(a) Every homomorphic binary reflexive relation in $\mathcal{V}$ is symmetric.

(b) Every homomorphic binary reflexive relation in $\mathcal{V}$ is transitive.

(c) Every homomorphic binary reflexive relation in $\mathcal{V}$ is an equivalence relation.

Note that the condition (c) above can be obtained either by combining the conditions (a) and (b), or directly from Theorem 2.4, by taking M in it to be a difunctionality matrix.

Now consider the case when $\mathcal{T} = \mathrm{Th}[\textit{pointed sets}]$ and M is the matrix

$$\left(\begin{array}{ccc|c} x & y & y & x \\ u & u & 0 & 0 \end{array}\right), \tag{3}$$

which is obtained from the difunctionality matrix, considered above, by replacing the two instances of v in it with 0. The corresponding system of term equations is

$$\begin{cases} p(x, y, y) = x, \\ p(u, u, 0) = 0. \end{cases}$$

As it was shown in [1], this system of equations is solvable in an algebraic theory $\mathcal{T}'$ with a nullary term 0, if and only if the system of equations

$$\begin{cases} s(x,0) = x, \\ s(u,u) = 0 \end{cases}$$

is solvable in $\mathcal{T}'$. This result can be also obtained from Theorem 3.2 and the following simple observation:

3.5. Lemma. *For a binary homomorphic relation R between pointed sets the following conditions are equivalent:*

(a) Both R and R^{op} are strictly closed with respect to the matrix (3).

(b) Both R and R^{op} are strictly closed with respect to the matrix

$$\left(\begin{array}{cc|c} x & 0 & x \\ u & u & 0 \end{array}\right). \tag{4}$$

For M equal to the matrix (3), or the matrix (4), $\mathbf{Var}_M$ is the class of subtractive varieties. The following characterization of subtractive varieties can be obtained in a similar way as the characterization 3.4 of Mal'tsev varieties.

3.6. Corollary (P. Agliano and A. Ursini [1]). *A variety $\mathcal{V}$ of universal algebras is subtractive (i.e.* $\mathrm{Th}[\mathcal{V}]$ *contains a nullary term* 0 *for which the systems of equations above are solvable) if and only if* $\mathrm{Th}[\mathcal{V}]$ *contains a nullary term* 0 *(which can be supposed to be the same as the nullary term involved in the equations) such that the following equivalent conditions are satisfied:*

(a) Every homomorphic binary reflexive relation in $\mathcal{V}$ is 0-symmetric.

(b) Every homomorphic binary reflexive relation in $\mathcal{V}$ is 0-transitive.

4. M-closed $\mathcal{T}$-enrichments

Let w and w' be terms of $\mathcal{T}$, with arities l and l', respectively. We say that w commutes with w' if

$$w(w'(x_{11},...,x_{1l'}),...,w'(x_{l1},...,x_{ll'})) = w'(w(x_{11},...,x_{l1}),...,w(x_{1l'},...,x_{ll'})). \tag{5}$$

We say that w is central if it commutes with every term of $\mathcal{T}$. In particular, any variable is a central term. A nullary term 0 is central if and only if $0 = 0'$ for any other nullary term $0'$. There exists a central nullary term in $\mathcal{T}$ if and only if $\mathbf{Alg}_{\mathcal{T}}$ is a pointed category, i.e. the terminal object in $\mathbf{Alg}_{\mathcal{T}}$ is at the same time initial.

An interpretation ι of $\mathcal{T}$ in an algebraic theory $\mathcal{T}'$ is said to be central if for every term w of $\mathcal{T}$, the corresponding term w^ι of $\mathcal{T}'$ is central in $\mathcal{T}'$.

We say that $\mathcal{T}$ is commutative if any of its terms is central. If $\mathcal{T}$ is commutative then the triple $(\mathbf{Alg}_{\mathcal{T}}, \otimes, \mathrm{Fr}_{\mathcal{T}}\{x\})$, where $\otimes$ denotes the tensor product of $\mathcal{T}$-algebras, is a closed monoidal category. A category enriched in $(\mathbf{Alg}_{\mathcal{T}}, \otimes, \mathrm{Fr}_{\mathcal{T}}\{x\})$ can be defined as a pair $(\mathcal{C}, H)$, where $\mathcal{C}$ is a category and H is a bifunctor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ such that the triangle

$$\begin{tikzcd} & \mathbf{Alg}_{\mathcal{T}} \arrow[d, "\text{forgetful functor}"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, bend left, "H"] \arrow[r, "\mathrm{hom}"'] & \mathbf{Set} \end{tikzcd} \tag{6}$$

commutes. Such a bifunctor H determines a functor $G : \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}\mathcal{C}$ (where $\mathbf{Alg}_{\mathcal{T}}\mathcal{C}$ denotes the category of internal $\mathcal{T}$-algebras in $\mathcal{C}$) for which the following triangle commutes:

$$\begin{tikzcd} & \mathbf{Alg}_{\mathcal{T}}\mathcal{C} \arrow[d, "\text{forgetful functor}"] \\ \mathcal{C} \arrow[ur, bend left, "G"] \arrow[r, "1_{\mathcal{C}}"'] & \mathcal{C} \end{tikzcd} \tag{7}$$

For each object A in $\mathcal{C}$ and for each l-ary term w in $\mathcal{T}$, the corresponding operation $w_{G(A)}$ in the $\mathcal{T}$-algebra structure of $G(A)$, is obtained as follows

$$w_{G(A)} = w_{H(A^l,A)}(\pi_1, ..., \pi_l) : A^l \longrightarrow A.$$

This correspondence

$$\{\text{Functors } H \text{ for which (6) commutes}\} \longrightarrow \{\text{Functors } G \text{ for which (7) commutes}\}$$

is a bijection. For any two objects A and B, and an l-ary term w, the operation $w_{H(A,B)}$ can be recovered from the operation $w_{G(B)}$ via the equality

$$w_{H(A,B)}(f_1, ..., f_l) = w_{G(B)} \circ (f_1, ..., f_l), \quad f_1, ..., f_l : A \longrightarrow B,$$

where $(f_1, ..., f_l)$ in the right hand side of the equality denotes the induced morphism $(f_1, ..., f_l) : A \longrightarrow B^l$.

We will call a bifunctor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$, for which (6) commutes, a $\mathcal{T}$-enrichment of $\mathcal{C}$, and we will denote the corresponding functor $\mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}\mathcal{C}$ by H^*.

Let H be a $\mathcal{T}$-enrichment of $\mathcal{C}$. Note that for any object X in $\mathcal{C}$, the functor $H(X, -) : \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ preserves limits (since $\mathrm{hom}(X, -)$ preserves limits, and the forgetful functor $\mathbf{Alg}_{\mathcal{T}} \longrightarrow \mathbf{Set}$ reflects them).

4.1. Definition. *A $\mathcal{T}$-enrichment H of a category $\mathcal{C}$ is said to be M-closed if the following equivalent conditions are satisfied:*

(a) For each object X in $\mathcal{C}$ the functor $H(X, -) : \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ is M-closed.

(b) The functor $H^* : \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}\mathcal{C}$ is M-closed.

Below we will give two characterizations of M-closed $\mathcal{T}$-enrichments (Proposition 4.2 and 4.3); the first one will be used to prove Theorem 4.4, and the second one will be used for Theorems 6.5 and 6.7.

4.2. PROPOSITION. *Let C be an object in a category $\mathcal{C}$ such that* $\hom(C,-) : \mathcal{C} \longrightarrow \mathbf{Set}$ *reflects isomorphisms. Then, a $\mathcal{T}$-enrichment H of $\mathcal{C}$ is M-closed if and only if $H(C,-)$ is M-closed.*

PROOF. This can be verified straightforwardly, using Proposition 2.2. ■

Let $R \longrightarrow A_1 \times ... \times A_n$ be an internal relation in $\mathcal{C}$, and X an object in $\mathcal{C}$. Consider an extended matrix

$$M' = \left(\begin{array}{ccc|c} f_{11} & \cdots & f_{1m} & g_1 \\ \vdots & & \vdots & \vdots \\ f_{n1} & \cdots & f_{nm} & g_n \end{array}\right)$$

whose each i-th row consists of morphisms $f_{i1}, ..., f_{im}, g_i : X \longrightarrow A_i$ of $\mathcal{C}$. We will say that the relation R is compatible with M' if the induced relation $\hom(X,R)$ between $\hom(X,A_1), ..., \hom(X,A_n)$ is compatible with M', that is, if whenever the morphisms $(f_{1j}, ..., f_{nj}) : X \longrightarrow A_1 \times ... \times A_n$ factor through r, so does the morphism $(g_1, ..., g_n) : X \longrightarrow A_1 \times ... \times A_n$.

4.3. PROPOSITION. *Suppose all terms in M depend only on one fixed variable (i.e. $k = 1$). Then, for any $\mathcal{T}$-enrichment H of any category $\mathcal{C}$, the following conditions are equivalent:*

(a) H is M-closed.

(b) Each relation $R \longrightarrow X^n$ in $\mathcal{C}$ is compatible with the extended matrix

$$\left(\begin{array}{ccc|c} (t_{11})_{H^*(X)} & \cdots & (t_{1m})_{H^*(X)} & (u_1)_{H^*(X)} \\ \vdots & & \vdots & \vdots \\ (t_{n1})_{H^*(X)} & \cdots & (t_{nm})_{H^*(X)} & (u_n)_{H^*(X)} \end{array}\right) \tag{8}$$

consisting of the unary operations $(t_{ij})_{H^*(X)}, (u_i)_{H^*(X)}$ of the $\mathcal{T}$-algebra $H^*(X)$.

PROOF. For any unary term w and for any object X in $\mathcal{C}$,

$$w_{H^*(X)} = w_{H(X,X)}(1_X).$$

This shows that the matrix (8) is a regular interpretation of M in $H(X,X)$, which gives (a) ⇒ (b).

(b) ⇒ (a): Suppose (b) is satisfied. We should show that for any relation $R \longrightarrow A^n$, for any object X, and for any morphism $f : X \longrightarrow A$, the relation R is compatible with the matrix

$$\left(\begin{array}{ccc|c} (t_{11})_{H(X,A)}(f) & \cdots & (t_{1m})_{H(X,A)}(f) & (u_1)_{H(X,A)}(f) \\ \vdots & & \vdots & \vdots \\ (t_{n1})_{H(X,A)}(f) & \cdots & (t_{nm})_{H(X,A)}(f) & (u_n)_{H(X,A)}(f) \end{array}\right).$$

For each unary term w,

$$w_{H(X,A)}(f) = w_{H^*(A)} \circ f = f \circ w_{H^*(X)}.$$

Thus, the extended matrix above is the same as the matrix

$$\left(\begin{array}{ccc|c} f\circ(t_{11})_{H^*(X)} & \cdots & f\circ(t_{1m})_{H^*(X)} & f\circ(u_1)_{H^*(X)} \\ \vdots & & \vdots & \vdots \\ f\circ(t_{n1})_{H^*(X)} & \cdots & f\circ(t_{nm})_{H^*(X)} & f\circ(u_n)_{H^*(X)} \end{array}\right).$$

Consider the relation $S \longrightarrow X^n$ obtained from R via the pullback

$$\begin{tikzcd} S \arrow[r] \arrow[d] & R \arrow[d] \\ X^n \arrow[r, "f\times\ldots\times f"'] & A^n \end{tikzcd}$$

By (b), S is compatible with (8). This implies that R is compatible with the matrix above. ∎

Let H be a $\mathcal{T}$-enrichment of a variety $\mathcal{V}$. For each term $w = w(x_1, ..., x_l)$ in $\mathcal{T}$, let $w^\iota = w_{H^*(\mathrm{Fr}_\mathcal{V}\mathcal{X})}(x_1, ..., x_l)$. This defines a central interpretation $\iota : \mathcal{T} \longrightarrow \mathrm{Th}[\mathcal{V}]$. Moreover, any central interpretation $\iota : \mathcal{T} \longrightarrow \mathrm{Th}[\mathcal{V}]$ can be defined in this way, using a unique $\mathcal{T}$-enrichment H of $\mathcal{V}$.

4.4. Theorem. *A $\mathcal{T}$-enrichment of a variety $\mathcal{V}$ is M-closed if and only if the corresponding central interpretation $\mathcal{T} \longrightarrow \mathrm{Th}[\mathcal{V}]$ is M-closed.*

Proof. Let H be a $\mathcal{T}$-enrichment of $\mathcal{V}$ and let ι be the corresponding central interpretation $\iota : \mathcal{T} \longrightarrow \mathrm{Th}[\mathcal{V}]$. The fact that ι is M-closed if and only if H is M-closed follows from Theorem 3.2, Proposition 4.2 and the following observations:

- The forgetful functor $\iota^* : \mathcal{V} \longrightarrow \mathbf{Alg}_\mathcal{T}$ is naturally isomorphic to the functor $H(\mathrm{Fr}_\mathcal{V}\{x\}, -) : \mathcal{V} \longrightarrow \mathbf{Alg}_\mathcal{T}$, which implies that ι^* is M-closed if and only if $H(\mathrm{Fr}_\mathcal{V}\{x\}, -)$ is M-closed,
- the functor $\hom(\mathrm{Fr}_\mathcal{V}\{x\}, -)$ reflects isomorphisms.

∎

5. Categories with M-closed relations

5.1. Definition. *A category $\mathcal{C}$ is said to have M-closed relations (or, it is said to be a category with M-closed relations) if any internal relation $R \longrightarrow A^n$ in $\mathcal{C}$ is M-closed with respect to any internal $\mathcal{T}$-algebra structure on A.*

If there exists a $\mathcal{T}$-enrichment of $\mathcal{C}$ we say that $\mathcal{C}$ is $\mathcal{T}$-enrichable. We are only interested in $\mathcal{T}$-enrichable categories with M-closed relations, and, further, we are only interested in the case when $\mathcal{T}$ is a commutative algebraic theory satisfying the following equivalent conditions:

- Any two $\mathcal{T}$-algebra structures on a set A, such that each operation w of the first $\mathcal{T}$-algebra structure commutes with every operation w' of the second $\mathcal{T}$-algebra structure (i.e. the equality (5) is satisfied for all $x_{ij} \in A$), necessarily coincide.
- For any $\mathcal{T}$-enrichment H on any category $\mathcal{C}$, and for any object A in $\mathcal{C}$, the $\mathcal{T}$-algebra structure of $H^*(A)$ is the unique internal $\mathcal{T}$-algebra structure on A.
- Any two $\mathcal{T}$-enrichments of the same category coincide.

In particular, the algebraic theories of sets and of pointed sets satisfy the above conditions.

5.2. Convention. *Henceforth we assume that $\mathcal{T}$ satisfies the above conditions.*

Then, in a $\mathcal{T}$-enrichable category, M-closedness and strict M-closedness of internal relations are uniquely determined — that is, say, M-closedness of a relation $R \longrightarrow A^n$ does not depend on the choice of an internal $\mathcal{T}$-algebra structure on A (as there is only one such structure).

5.3. Theorem. *For any $\mathcal{T}$-enrichable category $\mathcal{C}$, the following conditions are equivalent:*

(a) $\mathcal{C}$ has M-closed relations.

(b) The unique $\mathcal{T}$-enrichment H of $\mathcal{C}$ is M-closed.

Proof. (a) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (a) follows from the fact that for any object A in $\mathcal{C}$, the $\mathcal{T}$-algebra structure of $H^*(A)$ is the unique $\mathcal{T}$-algebra structure on A. ■

5.4. Corollary. *For any variety $\mathcal{V}$ of universal algebras, the following conditions are equivalent:*

(a) $\mathcal{V}$ is a $\mathcal{T}$-enrichable category with M-closed relations.

(b) There exists a central M-closed interpretation $\mathcal{T} \longrightarrow \mathrm{Th}[\mathcal{V}]$.

Note that in a $\mathcal{T}$-enrichable category $\mathcal{C}$, (strict) M-closedness is the same as (strict) (M, H^*)-closedness in the sense of Definition 2.3, where H is the unique $\mathcal{T}$-enrichment of $\mathcal{C}$. This observation allows us to derive the following characterization theorem for $\mathcal{T}$-enrichable categories with M-closed relations from the similar characterization (Theorem 2.4) of M-closed functors:

5.5. Theorem. *For any $\mathcal{T}$-enrichable category $\mathcal{C}$, the following conditions are equivalent:*

(a) Every relation $R \longrightarrow A_1 \times ... \times A_n$ in $\mathcal{C}$ is strictly M-closed.

(b) $\mathcal{C}$ has M-closed relations, i.e. every relation $R \longrightarrow A^n$ in $\mathcal{C}$ is M-closed.

If M satisfies the condition (*) from Section 1, then the conditions above are also equivalent to the following condition:

(c) Every reflexive relation $R \longrightarrow A^n$ in $\mathcal{C}$ is M-closed.

6. Closedness properties of internal relations in Mal'tsev, unital, strongly unital and subtractive categories

In the case when $\mathcal{T} = \mathrm{Th}[sets]$, any category $\mathcal{C}$ is $\mathcal{T}$-enrichable, and,

- the functor $\mathrm{hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Alg}_{\mathcal{T}}$ is the unique $\mathcal{T}$-enrichment of $\mathcal{C}$,
- and $\mathrm{hom}^* = 1_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C} = \mathbf{Alg}_{\mathcal{T}}\mathcal{C}$.

6.1. Definition (A. Carboni, M. C. Pediccho, and N. Pirovano [8]). *A category is said to be a Mal'tsev category if every binary reflexive relation in it is an equivalence relation.*[1]

To say that every (binary) reflexive relation in $\mathcal{C}$ is an equivalence relation is the same as to say that every reflexive relation in $\mathcal{C}$ is difunctional. Since the difunctionality matrix satisfies (**), by Theorem 5.5, reflexive relations in $\mathcal{C}$ are difunctional if and only if all binary relations are difunctional. Further, from Theorem 5.5 we can deduce the following characterization theorem for Mal'tsev categories, in the same way as the characterizations 3.3 and 3.4 of Mal'tsev varieties were deduced from Theorem 2.4:

6.2. Corollary (A. Carboni, M. C. Pediccho, and N. Pirovano [8]). *A category is a Mal'tsev category if and only if the following equivalent conditions are satisfied:*

(a) Every relation $r : R \longrightarrow A \times B$ is difunctional.

(b) Every relation $r : R \longrightarrow A \times A$ is difunctional.

(c) Every reflexive relation $r : R \longrightarrow A \times A$ is symmetric.

(d) Every reflexive relation $r : R \longrightarrow A \times A$ is transitive.

We also have the following result:

6.3. Theorem. *Let M be an arbitrary Mal'tsev matrix. Then, a category has M-closed relations if and only if it is a Mal'tsev category.*

Proof. This follows from the fact that strict closedness with respect to a Mal'tsev matrix is the same as difunctionality. ∎

[1] The notion of a Mal'tsev category was first introduced by A. Carboni, J. Lambek, and M. C. Pedicchio in [7]; however, in [7] the definition of a Mal'tsev category required in addition Barr exactness, which was later omitted in [8].

Henceforth $\mathcal{T} = \mathrm{Th}[\textit{pointed sets}]$. In this case a $\mathcal{T}$-enrichable category is the same as a pointed category. Recall that a pointed category is a category in which the terminal object is at the same time initial, and then it is called the zero object. Let $\mathcal{C}$ be a pointed category. The unique $\mathcal{T}$-enrichment H of $\mathcal{C}$ assigns to each pair of objects X and Y in $\mathcal{C}$ the pointed set $(\hom(X,Y),0)$, where the base point 0 is the unique morphism $0 : X \longrightarrow Y$ that factors through the zero object.

Recall that a pair of morphisms f, g having the same codomain C is said to be jointly extremal epimorphic, if any monomorphism $h : B \longrightarrow C$ such that both f and g factor through h is necessarily an isomorphism.

6.4. DEFINITION (D. BOURN [4], F. BORCEUX AND D. BOURN [2]). *A pointed category $\mathcal{C}$ is said to be unital if it satisfies the following equivalent conditions:*

(a) For any two objects A and B in $\mathcal{C}$, the product injections $\iota_1 = (1_A, 0) : A \longrightarrow A \times B$, $\iota_2 = (0, 1_B) : B \longrightarrow A \times B$ are jointly extremal epimorphic.

(b) For any object A in $\mathcal{C}$, the product injections $\iota_1, \iota_2 : A \longrightarrow A \times A$ are jointly extremal epimorphic.

(c) For any object A in $\mathcal{C}$, the morphisms $\iota_1, \iota_2, (1_A, 1_A) : A \longrightarrow A \times A$ are jointly extremal epimorphic.

6.5. THEOREM. *For any pointed category $\mathcal{C}$, the following conditions are equivalent:*

(a) $\mathcal{C}$ is unital.

(b) Every binary relation $r : R \longrightarrow A \times A$ in $\mathcal{C}$ is compatible with

$$\left(\begin{array}{cc|c} 1_A & 0 & 1_A \\ 0 & 1_A & 1_A \end{array}\right).$$

(c) $\mathcal{C}$ has M-closed relations, where

$$M = \left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right). \tag{9}$$

PROOF. (a) $\Rightarrow$ (b) is trivial.

(b) $\Leftrightarrow$ (c) follows from Proposition 4.3.

(c) $\Rightarrow$ (a) follows from the following simple observations:

- If (c) is satisfied then any binary relation $R \longrightarrow A \times B$ in $\mathcal{C}$ is strictly closed with respect to the matrix (9) (by Theorem 5.5),
- for any two objects A and B in $\mathcal{C}$, the matrix
$$\left(\begin{array}{cc|c} \pi_1 & 0 & \pi_1 \\ 0 & \pi_2 & \pi_2 \end{array}\right) \tag{10}$$
where π_1 and π_2 denote the product projections $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, is a row-wise interpretation of (9),

- $\mathcal{C}$ satisfies 6.4(a) if and only if any binary relation $R \longrightarrow A \times B$ in $\mathcal{C}$ is compatible with the matrix (10), for any two objects A and B in $\mathcal{C}$.

■

Below, by a pointed variety we mean a variety $\mathcal{V}$ of universal algebras, which is pointed as a category. For such $\mathcal{V}$, by 0 we denote the central nullary term of $\mathrm{Th}[\mathcal{V}]$.

From Theorem 6.5 and Corollary 5.4 we obtain:

6.6. Corollary (F. Borceux and D. Bourn [2]). *For any pointed variety $\mathcal{V}$, the following conditions are equivalent:*

(a) $\mathcal{V}$ is unital.

(b) $\mathcal{V}$ is a Jónsson-Tarski variety in the sense of J. D. H. Smith [19], i.e. the system of term equations

$$\begin{cases} u(x,0) = x, \\ u(0,x) = x \end{cases}$$

is solvable in $\mathrm{Th}[\mathcal{V}]$.

As defined in [12], a subtractive category is a pointed category in which for any reflexive relation $r : R \longrightarrow A \times A$, if $(0, 1_A) : A \to A \times A$ factors through r, then $(1_A, 0) : A \to A \times A$ also factors through r. Thus, a pointed category $\mathcal{C}$ is subtractive if every reflexive relation $R \longrightarrow A \times A$ in $\mathcal{C}$ is compatible with

$$\left(\begin{array}{c|c} 0 & 1_A \\ 1_A & 0 \end{array} \right),$$

or equivalently, if every relation $R \longrightarrow A \times A$ is compatible with

$$\left(\begin{array}{cc|c} 1_A & 0 & 1_A \\ 1_A & 1_A & 0 \end{array} \right).$$

From Proposition 4.3 we obtain:

6.7. Theorem. *A pointed category is subtractive if and only if it has M-closed relations, where*

$$M = \left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array} \right). \tag{11}$$

6.8. Corollary [12]. *For any pointed variety $\mathcal{V}$, the following conditions are equivalent:*

(a) $\mathcal{V}$ is a subtractive category.

(b) $\mathcal{V}$ is a subtractive variety in the sense of A. Ursini [22], i.e. the system of term equations

$$\begin{cases} s(x,0) = x, \\ s(x,x) = 0 \end{cases}$$

is solvable in $\mathrm{Th}[\mathcal{V}]$.

From Lemma 3.5 and Theorem 6.7 we obtain:

6.9. THEOREM. *A pointed category is subtractive if and only if it has M-closed relations, where*

$$M = \left(\begin{array}{ccc|c} x & y & y & x \\ x & x & 0 & 0 \end{array} \right).$$

For a pair of monomorphisms f, g having the same codomain, we will write $f \leqslant g$ if $f = gh$ for some morphism h. We write $f \approx g$ if $f \leqslant g$ and at the same time $g \leqslant f$, which is the case if and only if $f = gh$ for some isomorphism h.

Below $\mathrm{SEpis}\,\mathcal{C}$ denotes the class of all split epimorphisms in $\mathcal{C}$.

6.10. DEFINITION (D. BOURN [4],[5], F. BORCEUX AND D. BOURN [2]). *A pointed category $\mathcal{C}$ is said to be strongly unital if it satisfies the following equivalent conditions:*

(a) For any two objects A and B in $\mathcal{C}$ and for any relation $r = (r_1, r_2) : R \longrightarrow A \times B$,

$$[r_2 \in \mathrm{SEpis}\,\mathcal{C} \wedge [(1_A, 0) \leqslant r]] \Rightarrow 1_{A\times B} \approx r.$$

(b) For any object A in $\mathcal{C}$ and for any relation $r : R \longrightarrow A \times A$,

$$[[(1_A, 1_A) \leqslant r] \wedge [(1_A, 0) \leqslant r]] \Rightarrow 1_{A\times A} \approx r.$$

6.11. PROPOSITION [12]. *For any pointed category $\mathcal{C}$, the following conditions are equivalent:*

(a) $\mathcal{C}$ is strongly unital.

(b) $\mathcal{C}$ is subtractive and unital.

PROOF. $\mathcal{C}$ is subtractive if and only if

$$[[(1_A, 1_A) \leqslant r] \wedge [(1_A, 0) \leqslant r]] \Rightarrow (0, 1_A) \leqslant r$$

is satisfied for any relation $r : R \longrightarrow A \times A$ in $\mathcal{C}$. According to Definition 6.4(c), $\mathcal{C}$ is unital if and only if

$$[[(1_A, 1_A) \leqslant r] \wedge [(1_A, 0) \leqslant r] \wedge [(0, 1_A) \leqslant r]] \Rightarrow 1_{A\times A} \approx r$$

is satisfied for any $r : R \longrightarrow A \times A$. For each r, the implication in 6.10(b) is satisfied if and only if both of the implications above are satisfied. Hence, $\mathcal{C}$ is strongly unital if and only if $\mathcal{C}$ is subtractive and unital. ∎

6.12. THEOREM. *For any pointed category $\mathcal{C}$, the following conditions are equivalent:*

(a) $\mathcal{C}$ is strongly unital.

(b) $\mathcal{C}$ has M-closed relations, where

$$M = \left(\begin{array}{ccc|c} x & 0 & 0 & x \\ x & x & y & y \end{array} \right). \tag{12}$$

PROOF. This follows from Proposition 6.11 and the simple observation that a homomorphic binary relation between pointed sets is strictly closed with respect to the matrix (12) if and only if it is strictly closed with respect to both of the matrices (9) and (11), which determine the classes of unital and subtractive categories, respectively. ∎

6.13. Corollary (F. Borceux and D. Bourn [2]). *For any pointed variety $\mathcal{V}$, the following conditions are equivalent:*

(a) $\mathcal{V}$ *is strongly unital.*

(b) The system of term equations

$$\begin{cases} p(x,0,0)=x, \\ p(x,x,y)=y \end{cases}$$

is solvable in $\mathrm{Th}[\mathcal{V}]$.

References

[1] P. Agliano and A. Ursini, Ideals and other generalizations of congruence classes, *Journal of Australian Mathematical Society* (Series A) **53**, 1992, 103-115.

[2] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, *Mathematics and its applications* **566**, Kluwer 2004.

[3] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, Springer Lecture Notes in Mathematics **1488**, 1991, 43-62.

[4] D. Bourn, Mal'cev categories and fibration of pointed objects, *Applied Categorical structures* **4**, 1996, 307-327.

[5] D. Bourn, Intrinsic centrality and associated classifying properties, *Journal of Algebra* **256**, 2002, 126-145.

[6] A. Carboni, G. M. Kelly, and M. C. Pedicchio, Some remarks on Maltsev and Goursat Categories, *Applied Categorical Structures* **1**, 1993, 385-421.

[7] A. Carboni, J. Lambek, and M. C. Pedicchio, Diagram chasing in Mal'cev categories, *Journal of Pure and Applied Algebra* **69**, 1990, 271-284.

[8] A. Carboni, M. C. Pedicchio, and N. Pirovano, Internal graphs and internal groupoids in Mal'tsev categories, in Proc. Conf. Montreal 1991, 1992, 97-109.

[9] G. O. Findlay, Reflexive homomorphic relations, *Canadian Mathematical Bulletin* **3**, 1960, 131-132.

[10] P. Freyd and A. Scedrov, Categories, Allegories, North-Holland Mathematical Library **39**, North-Holland, 1990.

[11] Z. Janelidze, Matrices of terms and relations in categories, M.Sc. Thesis, Tbilisi State University, 2004.

[12] Z. Janelidze, Subtractive categories, *Applied Categorical Structures* **13**, No. 4, 2005, 343-350.

[13] Z. Janelidze, Normal product projections, subtractive categories, and elementary term conditions in universal and categorical algebra, Ph.D. Thesis, A. Razmadze Mathematical Institute of Georgian Academy of Sciences, 2006.

[14] Z. Janelidze, Closedness properties of internal relations II: Bourn localization, *Theory and Applications of Categories* **16**, 2006, 262-282.

[15] P. T. Johnstone, Affine categories and naturally Mal'tsev categories, *Journal of Pure and Applied Algebra* **61**, 1989, 251-256.

[16] B. Jónsson and A. Tarski, Direct decompositions of finite algebraic systems, Notre Dame Mathematical Lectures **5**, University of Notre Dame, Indiana, 1947.

[17] J. Lambek, Goursat's theorem and the Zassenhaus lemma, *Canadian Journal of Mathematics* **10**, 1957, 45-56.

[18] A. I. Mal'tsev, On the general theory of algebraic systems, *USSR Mathematical Sbornik N.S.* **35**, 1954, 3-20.

[19] J. D. H. Smith, Mal'cev varieties, Lecture Notes in Mathematics **554**, Springer, 1976.

[20] J. W. Snow, Maltsev conditions and relations on algebras, *Algebra Universalis* **42**, 1999, 299-309.

[21] A. Ursini, Osservazioni sulla varietá BIT, *Bolletino della Unione Matematica Italiana* **8**, 1973, 205-211.

[22] A. Ursini, On subtractive varieties, I, *Algebra Universalis* **31**, 1994, 204-222.

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