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EVERY GROUP IS REPRESENTABLE BY ALL NATURAL TRANSFORMATIONS OF SOME SET-FUNCTOR

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Abstract. For every group G, we construct a functor $F: \mathcal{SET} \to \mathcal{SET}$ (finitary for a finite group G) such that the monoid of all natural endotransformations of F is a group isomorphic to G.

1. Introduction

Classical results by G. Birkhoff [3] and J. de Groot [4] show that every group can be represented as the automorphism group of a distributive lattice, and as the automorphism group of a topological space. Since then many results of similar type were proven. An extensive survey about group-universality is presented in [5]. Many such representations are consequences of far more general results concerning representations of categories, see the monograph [9].

The structures, in which the group representation problem was considered, were always structured sets – algebras, topologies or relational structures. Our contribution is of a different nature, the structure being a set functor, i.e. an endofunctor of the category $\mathcal{SET}$ of all sets and mappings. Set functors were studied in late sixties and early seventies ([12, 13, 14, 6, 7, 8]) to get an insight into the behaviour of generalized algebraic categories $A(F, G)$. After about thirty years this field of problems has been refreshed and further results about internal structure of set functors obtained ([10, 11, 1, 2]).

The representation questions were not yet examined for set functors, the present paper is perhaps the first step in this direction. It solves the problem put to the authors by V. Trnková. We prove here a stronger form of the group representation problem: For every group G, we construct a set functor F such that every endotransformation of F is a natural equivalence and the group of all endotransformations of F is isomorphic to G. Moreover for a finite G, the functor F will be finitary.

1.1. Problem. *Can every monoid be respresented as the monoid of all endotransformations of a set functor? Is the quasicategory of all set functors even alg-universal or even universal? Recall that a (quasi)category $\mathcal{K}$ is alg-universal, if every category* $\mathrm{Alg}(\Sigma)$ *(the category of algebras with the signature* Σ *and algebra homomorphisms) can be fully embedded into* $\mathcal{K}$. $\mathcal{K}$ *is universal, if every concretizable category can be fully embedded into* $\mathcal{K}$.

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2. Notation

NOTATION FOR SETS AND MAPPINGS. Here $R \subseteq X, Y$ are sets, $x \in X$, $f\colon X \to Y$ is a mapping, $\sim$ is an equivalence on X:

α	Ordinal = set of smaller ordinals
κ, λ	Cardinal = set of smaller ordinals
$\lvert X\rvert$	Cardinality of X
$R \subset X$	R is a *proper* subset of X
id_X	Identity mapping $id_X\colon X \to X$
$\chi_{R,X}\colon X \to 2$	Characteristic mapping of R, i.e. $f(x) = 1$ iff $x \in R$
PX	Set of all subsets of X
$f[R]$	Image of R
$f^{-1}[R]$	Inverse image of R
$\mathrm{Im}(f)$	Image of X $(= f[X])$
$X/\sim$	Factor set X modulo $\sim$
$[x]_\sim$	Equivalence class of x modulo $\sim$

NOTATION FOR FUNCTORS. Here F is a set functor, X is a set, $x \in FX$, $\mu\colon F \to F$ is a natural transformation:

$\mathrm{Nat}(F)$	Monoid of all natural transformations of F
$\mathrm{NatEpi}(F)$	Monoid of all natural epitransformations of F
$\mu_X\colon FX \to FX$	X-th component of the transformation μ.
$\mathrm{Flt}(x)$	Filter of x (see the next section)
$\mathrm{Mon}(x)$	Monoid of x (see the next section)

NOTATION FOR GROUPS. Here $H \leq G$ are groups:

S_X	Symmetric group on X = group of all permutations on X.
$\mathrm{Aut}(G)$	Automorphism group of G
G/H	Factor group (if H is a normal subgroup of G)
$N_G H$	Normalizer of H in G, i.e. $N_G H = \{g \in G \mid gHg^{-1} = H\}$

3. Set functors

In this section, we recall some known facts about set functors, which will be needed in this paper.

Let $F : \mathcal{SET} \to \mathcal{SET}$ be a set functor. F is said to be **faithful**, if it is one-to-one on hom sets. Recall that F is faithful, if there exists an element $x \in F1$ such that $Fi_0(x) \neq Fi_1(x)$ for the two distinct mappings $i_0, i_1\colon 1 \to 2$ (see [12]).

F is said to be **connected**, if $|F1| = 1$.

Let F be a faithful connected set functor, X be a set, $x \in FX$. Then the set

$$\mathrm{Flt}(x) := \{U \mid U \subseteq X, (\exists u \in FU)\ Fi(u) = x \text{ where } i \text{ is the inclusion } i\colon U \to X\}$$

is a filter on X, we say **filter** of the point x (this is probably the most important property of set functors, see ([6])). F is said to be **finitary**, if there is a finite $R \in \mathrm{Flt}(x)$ for every

$x \in FX$.

Next important property of a point $x \in FX$ is its **monoid**:

$$\mathrm{Mon}(x) := \{f \mid f\colon X \to X,\ Ff(x) = x\}$$

The next three propositions are easy consequences of the definitions. They are known in more general forms, but we include the proofs for the reader's convenience:

3.1. Proposition. *Let F be a faithful connected set functor, X be a set, $x \in FX$, $U \in \mathrm{Flt}(x)$, $m \in \mathrm{Mon}(x)$. Then $m[U] \in \mathrm{Flt}(x)$ (and therefore $U \cap m[U] \in \mathrm{Flt}(x)$).*

Proof. Let $i\colon U \to X$, $j\colon m[U] \to X$ be the inclusions, $u \in U$ be such that $Fi(u) = x$, and $m'\colon U \to m[U]$ the restriction of m to U. Then $Fj(Fm'(u)) = Fm(Fi(u)) = x$ and $m[U] \in \mathrm{Flt}(x)$. ∎

3.2. Proposition. *Let F be a faithful connected set functor, $\mu\colon F \to F$ be a natural transformation, X be a set, $x \in FX$. Then*

1. $\mathrm{Flt}(x) \subseteq \mathrm{Flt}(\mu_X(x))$
2. $\mathrm{Mon}(x) \subseteq \mathrm{Mon}(\mu_X(x))$

Proof.

1. Let $U \in \mathrm{Flt}(x)$ and u such that $Fi(u) = x$. Then $Fi(\mu_U(u)) = \mu_X(Fi(u)) = \mu_X(x)$ and $U \in \mathrm{Flt}(\mu_X(x))$.
2. Let $m \in \mathrm{Mon}(x)$. Then $Fm(\mu_X(x)) = \mu_X(Fm(x)) = \mu_X(x)$ and $f \in \mathrm{Mon}(\mu_X(x))$.

∎

3.3. Proposition. *Let F be a faithful connected set functor, $\mu\colon F \to F$ be a natural transformation, $x \in F1$, X be a set, $f\colon 1 \to X$ be a mapping. Then $\mu_X(Ff(x)) = Ff(x)$.*

Proof. Clearly $\mu_X(Ff(x)) = Ff(\mu_1(x))$ and $\mu_1(x) = x$ from the connectedness. ∎

4. Constructions

We present here two constructions. In both of them, there figure some cardinal κ such that the constructed functor F has the following property: Every natural endotransformation $\mu : F \to F$, $\mu = \{\mu_X \mid X \text{ is a set }\}$ is completely determined by its κ-th component $\mu_\kappa\colon F\kappa \to F\kappa$. Then the submonoid

$$\mathrm{Nat}_\kappa(F) := \{\mu_\kappa \mid \mu \text{ is a natural endotransformation of } F\}$$

of the full transformation monoid on κ is isomorphic to the monoid of all natural endotransformations of F (the monoid operation is the composition of transformations).

A family of mappings μ is a natural endotransformation of F, iff $\mu_Y \circ Ff = Ff \circ \mu_X$ for every mapping $f : X \to Y$. The difference between the constructions is, which mappings f are used to force the monoid $\mathrm{Nat}_\kappa(F)$ to be isomorphic to a given group G. In the first one, which gives some partial results, the important mappings are bijections $f : \kappa \to \kappa$. In the second one, which gives the general result for groups of order at least 7, the important mappings are $f : \kappa \to 2$.

5. First construction – representing normalizers

In this section, we prove the following partial results:

5.1. Theorem. *Let G be a subgroup of S_κ, where $\kappa > 2$ is a finite cardinal. Then there exists a finitary set functor F such that* $\mathrm{Nat}(F) \cong N_{S_\kappa} G/G$

5.2. Theorem. *Let G be a subgroup of S_κ, where $\kappa > 2$ is a cardinal. Then there exists a set functor F such that* $\mathrm{NatEpi}(F) \cong N_{S_\kappa} G/G$

As an example, for the trivial subgroup $G = \{id_\kappa\}$ of S_κ, we obtain a representation of S_κ. There are also computed the normalizers in some other special cases (5.15, 5.16). This gives us representations of $\mathrm{Aut}(G)$ for arbitrary group G, $|G| > 2$ and also $\mathrm{Aut}(G)/H$, where H is a normal subgroup of $\mathrm{Aut}(G)$ such that $|G|$ and $|H|$ are co-prime natural numbers. In particular, this gives a representation of cyclic groups Z_{p^n} for arbitrary odd prime number p and natural number n, and representation of groups of order less than 7:

$$Z_2 \cong \mathrm{Aut}(Z_3), Z_3, Z_4 \cong \mathrm{Aut}(Z_5), Z_2^2 \cong \mathrm{Aut}(Z_8), Z_5, Z_6 \cong \mathrm{Aut}(Z_7), S_3.$$

To be complete, the trivial group Z_1 can be represented, for example, by the identity functor. On the other hand, the authors don't know the answer to the following question:

5.3. Problem. *Does there exist a group $G \leq S_\kappa$ such that $Q_8 \cong N_{S_\kappa} G/G$? Can κ be finite? Here Q_8 denotes the eight element quaternion group.*

Group structures. Group structures are used in the constructions both in this, where their role is essential, and the next section.

5.4. Definition. *Let G be a subgroup of S_κ, X be a set. Mappings $h, h' : \kappa \to X$ are said to be G-equivalent ($h \sim_G h'$), if there exists $g \in G$ such that $h' = hg$.*

5.5. Lemma. *The relation $\sim_G$ is an equivalence. Let $h \sim_G h' : \kappa \to X$ and $f : X \to Y$. Then* $\mathrm{Im}(h) = \mathrm{Im}(h')$ *and $fh \sim_G fh'$.*

Proof. Clearly, $\mathrm{Im}(hg) = \mathrm{Im}(h)$ for a bijection g; and $fh' = fhg$, if $h' = hg$. ■

5.6. Definition. *A pair $([h]_{\sim_G}, G)$, where $h{:}\kappa \to X$ is called a **G-structure** on X.*

The index $\sim_G$ will be omitted, because the equivalence is determined by G.

Construction. Let $\kappa > 2$ be a cardinal, G be a subgroup of S_κ. We are going to define a functor $F{:}\mathcal{SET} \to \mathcal{SET}$.

$$\begin{aligned} FX &= AX \coprod BX \\ AX &= \{R \mid \emptyset \neq R \subseteq X,\ |R| < \kappa\} \\ BX &= \{([h], G) \mid h{:}\kappa \to X,\ |\mathrm{Im}(h)| = \kappa,\ ([h], G) \text{ is a G-structure on } X\} \end{aligned}$$

FX is a disjoint union of AX and BX, i.e. $FX = AX \times \{0\} \cup BX \times \{1\}$, but the elements of FX will be written without the second component (for example $R \in FX$ rather than $(R, 0) \in FX$), since there is no danger of confusion.

For a mapping $f{:}X \to Y$ let

$$\begin{aligned} Ff(R) &= f[R] \\ Ff([h], G) &= \begin{cases} ([fh], G) & \text{if } |\mathrm{Im}(fh)| = \kappa \\ \mathrm{Im}(fh) & \text{otherwise} \end{cases} \end{aligned}$$

5.7. Corollary. *F is correctly defined faithful connected set functor.*

Proof. The definition is correct: Let $f{:}X \to Y$ be a mapping.

1. Let $\emptyset \neq R \subseteq X$. Then $\emptyset \neq f[R] \subseteq Y$, $|f[R]| < \kappa$, therefore $f[R] \in AY$.
2. Let $([h], G)$ be a G-structure on X and $|\mathrm{Im}(h)| = \kappa$. Then both $\mathrm{Im}(fh), [fh]$ don't depend on the choice of $h \in [h]$ (5.5). Either $|\mathrm{Im}(fh)| = \kappa$ or $|\mathrm{Im}(fh)| < \kappa$. In the first case $([fh], G) \in BY$. In the second case $\mathrm{Im}(fh) \in AY$.

F is a set functor: It should be checked that F preserve identities and composition. It's clear that $F(id_X) = id_{FX}$. Let $f{:}X \to Y$, $g{:}Y \to Z$.

1. Let $R \in AX$. Then $Fg(Ff(R)) = g[f[R]] = gf[R] = Fgf(R)$.
2. Let $([h], G) \in BX$.

   (a) $|\mathrm{Im}(fh)| = |\mathrm{Im}(gfh)| = \kappa$. Then $Fg(Ff([h], G)) = ([gfh], G) = Fgf([h], G)$.

   (b) $|\mathrm{Im}(gfh)| < |\mathrm{Im}(fh)| = \kappa$. Then $Fg(Ff([h], G)) = Fg([fh, G]) = \mathrm{Im}(gfh) = Fgf([h], G)$.

   (c) $|\mathrm{Im}(fh)| < \kappa$. Then $|\mathrm{Im}(gfh)| < \kappa$ and $Fg(Ff([h], G)) = \mathrm{Im}(gfh) = Fgf([h], G)$.

Since $B1 = \emptyset$ (because $\kappa > 1$) and $A1 = \{1\}$, the functor F is connected. For the two distinct mappings $i_0, i_1 : 1 \to 2$, $i_0(0) = 0$, we have $Fi_0(\{0\}) = \{0\} \neq \{1\} = Fi_1(\{0\})$, thus F is faithful. ∎

5.8. Corollary. *Let $R \in AX$, $([h], G) \in BX$, $U \subset X$. Then $U \in \mathrm{Flt}(R)$ iff $R \subseteq U$; and $U \in \mathrm{Flt}([h], G)$ iff $\mathrm{Im}(h) \subseteq U$. In particular, F is finitary for a finite κ.*

Proof. Let $U \in \mathrm{Flt}(R)$ an $i : U \to X$ be the inclusion. Then there exists some $S \in AU$ such that $R = Fi(S) = S \subseteq U$, because for any $([k], G) \in BU$ is $|\mathrm{Im}(ik)| = |\mathrm{Im}(k)| = \kappa$ and thus $Fi([k], G) \notin AX$.

Now let $U \in \mathrm{Flt}([h], G)$. Clearly there must be $([k], G) \in BU$ for which $h \sim_G ik$. Hence by (5.5) $\mathrm{Im}(h) = \mathrm{Im}(ik) = \mathrm{Im}(k) \in U$. ∎

Proof of Theorems 5.1 and 5.2. We will prove that $\mathrm{Nat}(F) \cong N_{S_\kappa}G/G$ for a finite κ and $\mathrm{NatEpi}(F) \cong N_{S_\kappa}G/G$ for an arbitrary κ. Let $\mu : F \to F$ be a natural transformation (or epitransformation for infinite κ).

5.9. Claim. *For every set X, μ_X is identical on AX, i.e. $\mu_X(R) = R$ for every $R \in AX$.*

Proof. Consider Y for which $|Y| < \kappa$ and thus $Y \in AY$. Then $S = \mu_Y(Y) \in AY$ since $BY = \emptyset$. Suppose $S \subset Y$ and take a bijective $f : Y \to Y$ such that $f[S] \neq S$. We have $f \in \mathrm{Mon}(Y)$, but $f \notin \mathrm{Mon}(S)$ a contradiction to (3.2). Thus $\mu_Y(Y) = Y$.

Now, for arbitrary $R \in AX$, we can find Y, $|Y| < \kappa$ and a mapping $f : Y \to X$ such that $\mathrm{Im}(f) = R$. Then $\mu_X(R) = \mu_X(Ff(Y)) = Ff(\mu_Y(Y)) = Ff(Y) = R$. ∎

5.10. Claim. *$\mu_\kappa([id_\kappa], G) \in B\kappa$.*

Proof. Suppose $\mu_\kappa([id_\kappa], G) = R \in A\kappa$. Since $R \subset \kappa$ (because $|R| < \kappa$), we can take $f : \kappa \to \kappa$ such that $|\mathrm{Im}(f)| < \kappa$ and $\mathrm{Im}(f) \neq f[R]$ since $\kappa > 2$. Then $Ff(\mu_\kappa([id_\kappa], G)) = Ff(R) = f[R]$, but $\mu_\kappa(Ff([id_\kappa], G)) = \mu_\kappa(\mathrm{Im}(f)) = \mathrm{Im}(f)$ (5.9). This contradicts the naturality of μ. ∎

Let $([k], G) = \mu_\kappa([id_\kappa], G)$, where $k : \kappa \to \kappa$.

5.11. Claim. *k is surjective.*

Proof. The proof is similar to that of 5.10: If $\mathrm{Im}(k) \subset \kappa$, we take $f : \kappa \to \kappa$ such that $|\mathrm{Im}(f)| < \kappa$ and $\mathrm{Im}(fk) \neq \mathrm{Im}(f)$. Then $Ff(\mu_\kappa([id_\kappa], G)) = \mathrm{Im}(fk) \neq \mathrm{Im}(f) = \mu_\kappa(Ff([id_\kappa], G))$. This is, again, a contradiction with the naturality of μ. ∎

5.12. Claim. *$\mu_X([h], G) = ([hk], G)$ for every $([h], G) \in BX$.*

Proof. We have $Fh(\mu_\kappa([id_\kappa], G)) = Fh([k], G) = ([hk], G)$ (because k is surjective) and $\mu_X(Fh([id_\kappa], G)) = \mu_X([h], G)$. ∎

5.13. Claim. *$k \in N_{S_\kappa}G$. μ is identical, if and only if $k \in G$.*

Proof. First, k is injective: For a finite κ, it is clear (because k is surjective). For infinite κ, suppose the contrary. Then $hk \not\sim_G id_\kappa$ for any $h : \kappa \to \kappa$ (because hk is not injective). This contradicts the assumption that μ is an epitransformation. Hence k is bijective (5.11).

$k \in N_{S_\kappa}G$: For every $g \in G$, we have $g \sim_G id_\kappa$ and therefore $gk \sim_G k$, because $([k], G) = \mu_\kappa([id_\kappa], G) = \mu_\kappa([g], G)) = ([gk], G)$ (5.12). Thus, there exists $g' \in G$ such that $gk = kg'$. This implies $k^{-1}gk \in G$ for every $g \in G$, hence $k \in N_{S_\kappa}G$.

If μ is identical, then in particular $([k], G) = \mu_\kappa([id_\kappa], G) = ([id_\kappa], G)$, thus $k \in G$.

Conversely, if $k \in G$, then $hk \sim_G h$ for every $([h], G) \in BX$. ∎

We have just proved that every (epi)transformation is of the form μ^k for some $k \in N_{S_\kappa}G$, where

$$\begin{aligned} \mu^k(R) &= R \\ \mu^k([h], G) &= ([hk], G) \end{aligned}$$

On the other hand, for every $k \in N_{S_\kappa}$, μ^k is defined correctly (independent of the choice of h) and it is a natural transformation of F : Let $f : X \to Y$ be a mapping.

1. $R \in AX$. Then $Ff(\mu^k_X(R)) = f[R] = \mu^k_Y(Ff(R))$.

2. $([h], G) \in BX$.

   (a) $|\mathrm{Im}(fh)| < \kappa$. Then $Ff(\mu^k_X([h], G)) = \mathrm{Im}(fhk) = \mathrm{Im}(fh) = \mu^k_Y(Ff([h], G)$.

   (b) $|\mathrm{Im}(fh)| = \kappa$. Then $Ff(\mu^k_X([h], G)) = ([fhk], G) = \mu^k_Y(Ff([h], G))$.

5.14. Claim. *The mapping $i : N_{S_\kappa}G \to \mathrm{Nat}(F)$ (resp. $i : N_{S_\kappa}G \to \mathrm{NatEpi}(F)$ for infinite κ) sending k to $\mu^{k^{-1}}$ is a group epimorphism,* $\mathrm{Ker}(i) = G$.

Proof. It is obvious that $\mu^{k'^{-1}} \circ \mu^{k^{-1}} = \mu^{(k'k)^{-1}}$, hence i is a grupoid homomorphism. The rest follows from 5.13. ∎

Now, the proof of 5.1 and 5.2 is complete. ∎

The following propositions are surely well known facts from the group theory. Therefore the proofs will be just outlined.

5.15. Proposition. *Let $G = (\kappa, \cdot)$ be a group and $l : G \to S_\kappa$ its left regular representation, let $G' := \mathrm{Im}(l)$. Then $N_{S_\kappa}G'/G' \cong \mathrm{Aut}(G)$.*

Proof. The largest group containing the regular group G' as a normal subgroup is the **holomorph** of G, i.e. subdirect product of G and $\mathrm{Aut}(G)$. Viewing $\mathrm{Aut}(G)$ as a subgroup of S_κ, this is simply $\mathrm{Aut}(G)G'$. ∎

5.16. Proposition. *Let κ be a finite cardinal. Let G, G', l be as in 5.15. Let H be a normal subgroup of* $\mathrm{Aut}(G)$ *such that $|H|$ and $|G| = \kappa$ are co-prime numbers. Let $K := G'H$. Then K is a group and $N_{S_\kappa}K/K \cong \mathrm{Aut}(G)/H$.*

Proof. Clearly K is normal subgroup of $\mathrm{Aut}(G)G'$. On the other hand, every inner automorphism of S_κ which fixes K fixes G' too, because G' contains exactly these elements of K, whose rank divides κ. Hence $N_{S_\kappa}(K)/K = N_{S_\kappa}(G')/K = \mathrm{Aut}(G)G'/HG' \cong \mathrm{Aut}(G)/H$. ■

Let $G = Z_{p^n}$, where p is an odd prime number and a n is a natural number. Then $\mathrm{Aut}(G) \cong Z_{(p-1)p^{n-1}}$. Let H be the subgroup of $\mathrm{Aut}(G)$ of order $p-1$. The proposition 5.16 gives us a representation of $\mathrm{Aut}(G)/H = Z_{p^{n-1}}$.

6. Second construction – main result

In this section, we prove the following:

6.1. Theorem. *Let G be a group of order at least 7. Then there exists a set functor F such that* $\mathrm{Nat}(F) \cong G$. *If G is of a finite order, then F is finitary.*

Groups of order less then 7 were represented in the previous section.

Let κ be a cardinal, $\kappa \geq 7$ and $G = (\kappa, \cdot)$ be a group.

Key proposition. Given an equivalence $\sim$ on some subset of $P\kappa$, let us denote

$$G_\sim := \{p \mid p{:}\kappa \to \kappa \text{ is a bijection}, \;\; R \sim p[R] \text{ for all } R \text{ where } \sim \text{ is defined } \}.$$

It's clear that $G_\sim$ is a subgroup of S_κ.

6.2. Proposition. *There exists an equivalence $\sim$ on $\kappa^{[2,3]}$ such that $G \cong G_\sim$. Here $\kappa^{[2,3]}$ denotes the set of all two and all three point subsets of κ.*

Let l_g denote the left translation $l_g{:}G \to G$, $l_g(h) = gh$. For $R, S \in G^{[2,3]}$ let

$$R \sim S \quad \text{iff} \quad S = l_g[R].$$

We are going to prove that $G_\sim = \{l_g \mid g \in G\}$, then $G \cong G_\sim$ follows.

It is clear that $l_g \in G_\sim$ for every $g \in G$. For the second inclusion, it suffices to prove that if $p \in G_\sim$, $p(1) = 1$ (1 is the unit element of G), then $p = id_\kappa$: If $q \in G_\sim$, then $l_{q(1)^{-1}} \circ q(1) = 1$ and we will have $l_{q(1)^{-1}} \circ q = id_\kappa$, thus $q = l_{q(1)}$.

Let $1 \neq g \in G$. If $\{1, g\} \sim \{1, h\}$, then clearly $h = g$ or $h = g^{-1}$. Thus either $p(g) = g$, or $p(g) = g^{-1}, p(g^{-1}) = g$, hence p consists only of fix points and transpositions. Suppose that p contains a transposition.

(1) Let $p = (1)(g\, g^{-1})(h)\ldots$ (this means that $1, g, g^{-1}, h$ are pairwise distinct, $p(h) = h$, $p(g) = g^{-1}$, $p(g^{-1}) = g$).

We prove that $g^4 = 1, h = g^2 (= g^{-2})$. Because $\{1, g, h\} \sim \{1, g^{-1}, h\}$, there exists $a \in G$ such that $\{a, ag, ah\} = \{1, g^{-1}, h\}$. We have three possibilities $a = 1, ag = 1, ah = 1$.

(a) $a = 1 \dots$ then $g = g^{-1}$ a contradiction.

(b) $a = g^{-1} \dots \{g^{-1}, 1, g^{-1}h\} = \{1, g^{-1}, h\}$. Then $g^{-1}h = h$, hence $g = 1$, a contradiction.

(c) $a = h^{-1} \dots \{h^{-1}, h^{-1}g, 1\} = \{1, g^{-1}, h\}$. Then $h^{-1} = h$ and $h^{-1}g = g^{-1}$ (it is not possible that $h^{-1} = g^{-1}$) and we have $h = g^2, h^2 = 1$.

(2) Let $p = (1)(g\ g^{-1})(h\ h^{-1})$. We prove that $g^5 = h^5 = 1, \{g^2, g^{-2}\} = \{h, h^{-1}\}$. Because $\{1, g, h\} \sim \{1, g^{-1}, h^{-1}\}$, there exists $a \in G$ such that $\{a, ag, ah\} = \{1, g^{-1}, h^{-1}\}$. There are three possibilities:

(a) $a = 1 \dots \{g, h\} = \{g^{-1}, h^{-1}\}$. Either $g = g^{-1}$ or $g = h^{-1}$, a contradiction.

(b) $a = g^{-1} \dots g^{-1}h = h^{-1}$, hence $g = h^2$.

(c) $a = h^{-1} \dots h^{-1}g = g^{-1}$, hence $h = g^2$.

Without lost of generality, assume $h = g^2$. Because $\{g, g^{-1}, h\} \sim \{g, g^{-1}, h^{-1}\}$, there exists $a \in G$ such that $\{ag, ag^{-1}, ag^2\} = \{g, g^{-1}, g^{-2}\}$.

(a) $a = 1 \dots g^2 = g^{-2}$. Then $h^{-1} = g^{-2} = g^2 = h$, a contradiction.

(b) $a = g^{-2} \dots \{g^{-3}, 1\} = \{g, g^{-2}\}$. Either $1 = g$, or $1 = g^{-2}$, a contradiction.

(c) $a = g^{-3} \dots g^{-4} = g$. Then $g^5 = 1$.

Together $g^5 = 1$ and $h = g^2$.

From (1) it follows that there is at most one fix point other than 1. From (2) it follows that there are at most two distinct transpositions. This contradicts $|G| \geq 7$. In fact, this works also for $|G| = 6$. ■

r-structures and s-structures. In the construction, we will need two equivalences – an equivalence $\approx$ on $P\kappa$ such that $G_{\approx} \cong G$ and an equivalence on $P\lambda$ (λ is a cardinal) with some properties. It will be more convenient to work with the quotient mappings rather then with the equivalences.

6.3. Lemma. *There exist a set E, a cardinal $\lambda > \kappa + \kappa$ (finite for finite κ) and mappings $q: P\kappa \to E$, $r: P\lambda \to E$ with the following properties:*

(Q1) $G \cong G_q = \{k \mid k: \kappa \to \kappa \text{ a bijection, } (\forall R \subseteq \kappa)\ q(k[R]) = q(R)\}$

(Q2) $q(\emptyset) \neq q(R)$, *if* $\emptyset \neq R \subseteq \kappa$

(Q3) $q(\kappa) \neq q(R)$, *if* $R \subset \kappa$

(Q4) $q(R) \neq q(S)$, *if* $|R| = 1$, $|S| > 1$, $|\kappa - S| > 1$

(R1) r *is onto* E

(R2) $r(\emptyset) = r(R)$, *iff* $|R| \leq 1$

(R3) $r(R) = r(b[R])$, *if* $R \subseteq \lambda$, $b : \lambda \to \lambda$ *is a bijection*

(R4) *If* $r(R) = r(S)$, *then* $r(\lambda - R) = r(\lambda - S)$ *for every* $R, S \subseteq \lambda$

(QR1) $q(\emptyset) = r(\emptyset)$, $q(\kappa) = r(\lambda)$

(QR2) *If* $q(R) = r(S)$, *then* $q(\kappa - R) = r(\lambda - S)$, *where* $R \subseteq \kappa$, $S \subseteq \lambda$

(QR3) *If* $q(R) = r(S)$, *where* $|R| = 1$, *then either* $|S| = 2$ *or* $|\lambda - S| = 2$

PROOF. Let $\sim$ be the equivalence from 6.2. We enumerate the set $(\kappa^{[2,3]}/\sim)$ by ordinals greater than three and add four elements:

$$E := (\kappa^{[2,3]}/\sim) \cup \{e_0, e_1, e_2, e_3\} = \{e_i \mid i \in \alpha\}.$$

Let $q : P\kappa \to E$ be the mapping

$$q(R) = \begin{cases} e_0 & \text{if } R = \emptyset \\ e_1 & \text{if } R = \kappa \\ e_2 & \text{if } |R| = 1 \text{ or } |\kappa - R| = 1 \\ [R]_\sim & \text{if } |R| \in \{2,3\} \\ [\kappa - R]_\sim & \text{if } |\kappa - R| \in \{2,3\} \\ e_3 & \text{otherwise,} \end{cases}$$

where $R \subseteq \kappa$. The definition of q is clearly correct (the cases are disjoint for $\kappa \geq 7$) and $G_q = G_\sim \cong G$.

Let λ_i denote the i-th cardinal number (i.e. $\lambda_0 = 0, \lambda_1 = 1, \ldots$). Let $\lambda > \lambda_{2\cdot\alpha}$, $\lambda > \kappa + \kappa$ be a cardinal (resp. a finite cardinal for finite κ). We define $r : P\lambda \to E$ as follows:

$$r(R) = \begin{cases} e_0 & \text{if } |R| \leq 1 \\ e_1 & \text{if } |\lambda - R| \leq 1 \\ e_i & \text{if } |R| = \lambda_i \text{ or } |\lambda - R| = \lambda_i, \text{ where } 2 \leq i < \alpha \\ e_3 & \text{otherwise,} \end{cases}$$

where $R \subseteq \lambda$. It is easy to see that all required properties are satisfied. ■

6.4. DEFINITION. *Let* X *be a set,* $s : PX \to E$ *be a mapping. We say that* s *is a* ***q-structure*** *on* X*, if there exists a mapping* $f : \kappa \to X$ *such that* $|\mathrm{Im}(f)| < \kappa$ *and* $s = qf^{-1}$.

We say that s *is an* ***r-structure*** *on* X*, if there exists a mapping* $f : \lambda \to X$ *such that* $|\mathrm{Im}(f)| < \lambda$ *and* $s = rf^{-1}$.

6.5. LEMMA. *Let* rf^{-1} *be an* r*-structure on* X *and* $g : \lambda \to \lambda$ *be a bijection. Then* $rf^{-1} = rg^{-1}f^{-1}$.

PROOF. It is an easy consequence of (R3). ■

6.6. Lemma. *q is neither a q-structure nor an r-structure.*

Proof. q is not a q-structure: If $f:\kappa \to \kappa$ is a mapping with $R = \mathrm{Im}(f), |R| < \kappa$, then $qf^{-1}[R] = q(\kappa) \neq q(R)$ (Q3).

q is not an r-structure: Suppose that $f : \lambda \to \kappa$ is a mapping such that $rf^{-1} = q$. For $a \in \kappa$, put $I_a := f^{-1}[\{a\}]$. Either $|I_a| = 2$ or $|\lambda - I_a| = 2$ for every $a \in \kappa$ (because $r(I_a) = q(\{a\})$ and (QR3)). If the second possibility occurs for some $a \in \kappa$ neither of the possibilities can hold for the rest of κ because $\kappa > 2$. Thus $\{I_a \mid a \in \kappa\}$ is a partition of λ to κ two-point sets, which is impossible, since $\lambda > \kappa + \kappa$. ■

The r-structures on 2 are in one-to-one correspondence with elements of E:

6.7. Lemma. *Let $R, S \subseteq \lambda$. Then $r\chi^{-1}_{R,\lambda} = r\chi^{-1}_{S,\lambda}$ iff $r(R) = r(S)$.*

Proof. For every $T \subseteq \lambda$, we have $r\chi^{-1}_{T,\lambda}[\emptyset] = r(\emptyset)$, $r\chi^{-1}_{T,\lambda}[\{0,1\}] = r(\lambda)$, $r\chi^{-1}_{T,\lambda}[\{0\}] = r(\lambda - T)$, $r\chi^{-1}_{T,\lambda}[\{1\}] = r(T)$. Thus, by (R4) $r\chi^{-1}_{R,\lambda} = r\chi^{-1}_{S,\lambda}$ iff $r(R) = r(S)$. ■

6.8. Lemma. *Every q-structure on 2 is an r-structure.*

Proof. Every mapping $f:\kappa \to 2$ is a characteristic mapping of some $R \subseteq \kappa$. Let $S \subseteq \lambda$ be such that $r(S) = q(R)$ (it exists due to (R1)). Then $q\chi^{-1}_{R,\kappa} = r\chi^{-1}_{S,\lambda}$: $q\chi^{-1}_{R,X}[\emptyset] = q(\emptyset) = r(\emptyset) = r\chi^{-1}_{S,X}[\emptyset]$ (due to (QR1)), $q\chi^{-1}_{R,X}[\{0,1\}] = q(\kappa) = r(\lambda) = r\chi^{-1}_{S,\lambda}[\{0,1\}]$ (QR1), $q\chi^{-1}_{R,X}[\{1\}] = q(R) = r(S) = r\chi^{-1}_{S,\lambda}[\{1\}]$, and finally $q\chi^{-1}_{R,X}[\{0\}] = q(\kappa - R) = r(\lambda - S) = r\chi^{-1}_{S,\lambda}[\{0\}]$ (QR2). ■

The converse (ie. every r-structure on 2 is a q-structure) is also true, but we will not need this fact (except for the motivation below the construction).

Construction. For a set X, we define

$$\begin{aligned} FX &= AX \coprod BX \coprod CX \\ AX &= \{s \mid s \text{ is a } q\text{-structure or an } r\text{-structure on } X\} \\ BX &= \{h \mid h:\kappa \to X, |\mathrm{Im}(h)| = \kappa\} \\ CX &= \{([h], S_\lambda) \mid h:\lambda \to X, |\mathrm{Im}(h)| = \lambda, ([h], S_\lambda) \text{ is a } S_\lambda\text{-structure}\} \end{aligned}$$

Again, the elements of the coproduct will be written without the second component.

For a mapping $f:X \to Y$, put

$$\begin{aligned} Ff(s) &= sf^{-1} \\ Ff(h) &= \begin{cases} fh & \text{if } |\mathrm{Im}(fh)| = \kappa \\ qh^{-1}f^{-1} & \text{otherwise} \end{cases} \\ Ff([h], S_\lambda) &= \begin{cases} ([fh], S_\lambda) & \text{if } |\mathrm{Im}(fh)| = \lambda \\ rh^{-1}f^{-1} & \text{otherwise} \end{cases} \end{aligned}$$

Before we start to prove Theorem 6.1, let us try to explain the roles of the components to give the reader better insight into the construction.

The "most important" parts of the functor F are $A2$ and $B\kappa$: We have proved (in 6.7, 6.8) that r-structures (and q-structures) on 2 are in one-to-one correspondence with the elements of E (i.e. with equivalence classes of $\mathrm{Ker}(q)$ and $\mathrm{Ker}(r)$). Let μ be an endotransformation of F such that $\mu_2 = id_{F2}$ and $\mu_\kappa(id_\kappa) = k$, where $k:\kappa \to \kappa \in B\kappa$ is a bijection. Let R be a subset of κ. The naturality of μ for the element $id_\kappa \in B\kappa$ gives us

$$\mu_2(F\chi_{R,\kappa}(id_\kappa)) = F\chi_{R,\kappa}(\mu_\kappa(id_\kappa)).$$

The left side equals $q\chi^{-1}_{R,\kappa}$ which corresponds to $q(R)$ under the correspondence mentioned above. The right side equals $q\chi^{-1}_{k^{-1}[R],\kappa}$ which corresponds to $q(k^{-1}[R])$. We see that k must be in the group G_q. The component B and q-structures are there to make a set endofunctor from this idea. The component C and r-structures are used to ensure that every natural transformation is an equivalence and that μ_2 is the identity.

6.9. Corollary. *F is correctly defined faithful connected set functor.*

Proof. The definition is correct: Let $f:X \to Y$ be a mapping.

1. Let s be a q-structure, i.e. $s = qg^{-1}$ for some $g:\kappa \to 2$, $|\mathrm{Im}(g)| < \kappa$. Then $|\mathrm{Im}(fg)| < \kappa$ and $qg^{-1}f^{-1} = q(fg)^{-1}$ is a q-structure. The argument for r-structures is analogical.
2. Let h be a mapping such that $|\mathrm{Im}(h)| = \kappa$. Either $|\mathrm{Im}(fh)| = \kappa$ hence $Ff(h) = fh \in BY$, or $|\mathrm{Im}(fh)| < \kappa$ hence $Ff(h) = qh^{-1}f^{-1} \in AY$.
3. Let $([h], S_\lambda)$ be a S_λ-structure, $|\mathrm{Im}(h)| = \lambda$. Then $\mathrm{Im}(fh), [fh], rh^{-1}f^{-1}$ do not depend on the choice of $h \in [h]$ (5.5, 6.5). Either $|\mathrm{Im}(fh)| = \lambda$ or $|\mathrm{Im}(fh)| < \lambda$. In the first case $([fh], S_\lambda) \in CY$. In the second case $rh^{-1}f^{-1} \in AY$.

F is a set functor: It should be checked that F preserve identities and composition. It's clear that $F(id_X) = id_{FX}$. Let $f:X \to Y$, $g:Y \to Z$ be mappings.

1. Let s be a s-structure or an r-structure. Then $Fg(Ff(x)) = sf^{-1}g^{-1} = s(gf)^{-1} = Fgf(x)$.
2. Let h be a mapping $h:\kappa \to X$.
   (a) $|\mathrm{Im}(fh)| = \kappa$, $|\mathrm{Im}(gfh)| = \kappa$. Then $Fg(Ff(x)) = gfh = Fgf(x)$.
   (b) $|\mathrm{Im}(fh)| = \kappa$, $|\mathrm{Im}(gfh)| < \kappa$. Then $Fg(Ff(x)) = Fg(fh) = qh^{-1}f^{-1}g^{-1} = Fgf(x)$.
   (c) $|\mathrm{Im}(fh)| < \kappa$. Then $|\mathrm{Im}(gfh)| < \kappa$ and $Fg(Ff(x)) = Fg(qh^{-1}f^{-1}) = qh^{-1}f^{-1}g^{-1} = Fgf(x)$.

3. Let $([h], S_\lambda)$ be a S_λ-structure. The proof is similar to that of 2.

Let $c_\kappa : \kappa \to 1$ and $c_\lambda : \lambda \to 1$ be the unique mappings. Since $qc_\kappa^{-1} = rc_\lambda^{-1}$ (the value at $\emptyset$ is $q(\emptyset) = r(\emptyset)$ (QR1), the value at $\{0\}$ is $q(\kappa) = r(\lambda)$ (QR1)) and $B1 = C1 = \emptyset$ ($\kappa, \lambda > 1$), the functor F is connected.

For the two distinct mappings $i_0, i_1 : 1 \to 2$, $i_0(0) = 0$, we have $Fi_0(rc_\kappa^{-1}) = rc_\lambda^{-1} i_0^{-1} = r\chi_{\emptyset,\lambda}^{-1} \neq r\chi_{\lambda,\lambda}^{-1} = rc_\lambda^{-1} i_1^{-1} = Fi_1(rc^{-1}{}_\lambda))$, because $r(\lambda) \neq r(\emptyset)$ ((QR1), (Q3), 6.7). Thus F is faithful. ∎

6.10. Corollary. *$f[\kappa] \in \mathrm{Flt}(qf^{-1})$, $f[\lambda] \in \mathrm{Flt}(rf^{-1})$; $S \in \mathrm{Flt}(h)$ iff $\mathrm{Im}(h) \subseteq S$; $S \in \mathrm{Flt}([h], S_\lambda)$ iff $\mathrm{Im}(h) \subseteq S$. In particular, if κ is finite, then F is finitary.*

Proof. Let $i : f[\kappa] \to X$ be the inclusion and $f' : \kappa \to f[\kappa]$ the restriction of f to the image. Then clearly $qf^{-1} = qf'^{-1} i^{-1} = Fi(qf')$. Similarly for an r-structure rf^{-1} and a mapping h. The last two statements can be proved in the same way as (5.8). ∎

Proof of Theorem 6.1. We will show that $\mathrm{Nat}(F) \cong G_q$. Similarly as in the proof of 5.1, 5.2, we will show in series of claims, that every natural transformation $\mu : F \to F$ is of the form μ^k, where k is a bijection, $k \in G_q$ (see the end of the proof for the definition of μ^k). This will provide us an isomorphism $i : G_q \to \mathrm{Nat}(F)$.

So, let $\mu : F \to F$ be a natural transformation.

6.11. Claim. *Let $R \subseteq \lambda$, $|R| \leq 1$. Then $\mu_2(r\chi_{R,\lambda}^{-1}) = r\chi_{R,\lambda}^{-1}$.*

Proof. We have observed in the proof of 6.9 that $Fi_0(x) = r\chi_{\emptyset,\lambda}^{-1}$ for the point $x \in F1$ and the mapping $i_0 : 1 \to 2$, $i_0(0) = 0$. Since $r\chi_{R,\lambda}^{-1} = r\chi_{\emptyset,\lambda}^{-1}$ ((R2) and 6.7), the statement follows from 3.3. ∎

6.12. Claim. *$\mu_\lambda([id_\lambda], S_\lambda) = ([id_\lambda], S_\lambda)$.*

Proof. Every bijective $f : \lambda \to \lambda$ is in the monoid of $([id_\lambda], S_\lambda)$, therefore is in the monoid of $\mu_\lambda([id_\lambda], S_\lambda)$ (3.2.2). Thus, if $S \in \mathrm{Flt}(\mu_\lambda([id_\lambda], S_\lambda))$, then $f[S] \cap S \in \mathrm{Flt}(\mu_\lambda([id_\lambda], S_\lambda))$ (3.1). If $|S| < \lambda$ we can find either a bijection f such that $f[S] \cap S = \emptyset$ or, for finite λ, a finite sequence of bijections $f_1, \ldots, f_n$ satisfying $f_1[S] \cap \ldots \cap f_n[S] \cap S = \emptyset$, both cases leading to a contradiction. Hence by (6.10) $\mu_\lambda([id_\lambda, S_\lambda]) = ([h], S_\lambda) \in C_\lambda$ and the same argument as in (5.11) gives us that h is a surjection.

Now since every bijection is equivalent to id_λ modulo $\sim_{S_\lambda}$, it suffices to show that h is injective. If not, let $a \in \lambda$ be such that $|h^{-1}[\{a\}] > 1$. From the naturality of μ, $\mu_2(F\chi_{\{a\},\lambda}([id_\lambda, S_\lambda])) = \mu_2(r\chi_{\{a\},\lambda}^{-1}) = r\chi_{\{a\},\lambda}^{-1}$ (6.11) should be equal to

$$F\chi_{\{a\},\lambda}(\mu_\lambda([id_\lambda, S_\lambda])) = F\chi_{\{a\},\lambda}([h, S_\lambda]) = rh^{-1}\chi_{\{a\},\lambda}^{-1} = r\chi_{h^{-1}[\{a\}],\lambda}^{-1},$$

a contradiction (6.7), (R2). ∎

6.13. Claim. *μ_X is identical on $AX \cup CX$, in particular $\mu_2 = id_{F2}$.*

Proof. μ_X is identical on CX and r-structures: Let $h : \lambda \to X$ be a mapping such that $|\mathrm{Im}(h)| = \lambda$. Then $\mu_X([h], S_\lambda) = \mu_X(Fh([id_\lambda], S_\lambda)) = Fh(\mu_\lambda([id_\lambda], S_\lambda)) = Fh([id_\lambda], S_\lambda) = ([h], S_\lambda)$. For a mapping $h : \lambda \to X$ such that $|\mathrm{Im}(h)| < \lambda$, the same computation gives $\mu_X(s) = s$ for every r-structure s on X.

$\mu_2 = id_{F2}$: Let $x \in F2$. $B2 = \emptyset$ and $C2 = \emptyset$, because $\kappa, \lambda > 2$. Hence $x \in A2$. But every q-structure on 2 is an r-structure (6.8).

μ_X is identical on AX: Let s be a q-structure. The filter of s contains a set of cardinality less than κ (6.10), hence the filter of $\mu_X(s)$ contains a set of cardinality less than κ (3.2.1). Thus $\mu_X(s)$ is a q-structure or an r-structure (6.10 again). For every $f : X \to 2$, we have $\mu_2(Ff(s)) = \mu_2(sf^{-1}) = sf^{-1}$ (we have used $\mu_2 = id_{F2}$) and $Ff(\mu_X(s)) = (\mu_X(s))f^{-1}$. Putting $f = \chi_{R,X}$ and computing the value of sf^{-1} and $(\mu_X(s))f^{-1}$ in $\{1\}$, we obtain $s(R) = (\mu_X(s))(R)$ for every $R \subseteq X$. ∎

6.14. Claim. *$\mu_\kappa(id_\kappa) \in B\kappa$.*

Proof. If $\mu_\kappa(id_\kappa) = s$ is a q-structure or r-structure, then $Ff(\mu_\kappa(id_\kappa)) = sf^{-1}$ and $\mu_2(Ff(id_\kappa)) = \mu_2(qf^{-1}) = qf^{-1}$ for every $f : \kappa \to 2$. Thus $s = q$ (see the end of the proof of 6.13 for details), but q is neither q-structure not r-structure (6.6), a contradiction.

If $\mu_\kappa(id_\kappa) = ([h], S_\lambda)$ is a S_λ-structure, then $\kappa \in \mathrm{Flt}(id_\kappa)$, but $\kappa \notin \mathrm{Flt}([h], S_\lambda))$ (6.10). This contradicts 3.2.1. ∎

Let $k = \mu_\kappa(id_\kappa)$, where $k : \kappa \to \kappa$, $|\mathrm{Im}(\kappa)| = \kappa$.

6.15. Claim. *$\mu_X(h) = hk$ for every $h \in BX$, $k \in G_q$.*

Proof. $\mu_X(h) = hk$: $\mu_X(h) = \mu_X(Fh(id_\kappa)) = Fh(\mu_\kappa(id_\kappa) = Fh(k) = hk$.

For arbitrary $f : \kappa \to 2$, we have $qk^{-1}f^{-1} = Ffk(id_\kappa) = Ff(\mu_\kappa(id_\kappa)) = \mu_2(Ff(id_\kappa)) = Ff(id_\kappa) = qf^{-1}$ hence $q = qk^{-1}$ (6.13).

Now, it suffices to prove that k is bijective, then clearly $k \in G_q$.

If k is not surjective, then there exists $R \subseteq \kappa$ such that $|R| = 1$, $k^{-1}[R] = \emptyset$. But $q(R) \neq q(k^{-1}[R])$ (Q2) a contradiction.

If k is not injective, then there exists $R \subseteq \kappa$ such that $|R| = 1$ and $|k^{-1}[R]| > 1$. Then clearly $|\kappa - k^{-1}[R]| > 1$ (because k is surjective and $\kappa > 2$). But $q(R) \neq q(k^{-1}[R])$ (Q4), a contradiction. ∎

We have just proved that every natural transformation is of the form μ^k for some $k \in G_q$, where

$$\begin{aligned} \mu^k(s) &= s \\ \mu^k([h], S_\lambda) &= ([h], S_\lambda) \\ \mu^k(h) &= hk \end{aligned}$$

On the other hand, this is a natural transformation of F for every $k \in G_\kappa$:

Let $f : X \to Y$ be a mapping.

1. $x \in AX \cup CX$. Then also $Ff(x) \in AY \cup CY$ and there is nothing to verify.

2. $([h], G) \in BX$.

   (a) $|\mathrm{Im}(fh)| < \kappa$. Then $Ff(\mu_X^k(h)) = qk^{-1}f^{-1} = qf^{-1} = \mu_Y^k(Ff(h))$, because $q = qk^{-1}$ for every $k \in G_q$.

   (b) $|\mathrm{Im}(fh)| = \kappa$. Then $Ff(\mu_X^k(h)) = fhk = \mu_Y^k(Ff(h))$.

6.16. Claim. *The mapping* $i : G_q \to \mathrm{Nat}(F)$ *sending* k *to* $\mu^{k^{-1}}$ *is a group isomorphism.*

This finishes the proof of 6.1. ■

6.17. Remark. The assumption $\kappa \geq 7$ is not essential in this construction. The key proposition 6.2 can be improved to cover the small cases. The reasons to include the first construction were:

- It answered the first question that the authors considered: Is it possible to represent Z_3?
- The proof of 5.1 and 5.2 is easier and contains some methods, which are used in the proof that the second construction (more involved) works.
- It is based on the observation, how the monoids of points of a set functor F affect the monoid $\mathrm{Nat}(F)$. It could be useful, when one wants to compute the monoid $\mathrm{Nat}(F)$ for a given set functor F.

References

[1] A. Barkhudaryan, R. El Bashir, and V. Trnková. Endofunctors of *Set. in: Proceedings of the Conference Categorical Methods in Algebra and Topology, Bremen 2000, eds: H. Herrlich and H.-E. Porst, Mathematik-Arbeitspapiere*, 54:47–55, 2000.

[2] A. Barkhudaryan, R. El Bashir, and V. Trnková. Endofunctors of *Set* and cardinalities. *Cahiers Topo. Geom. Diff. Cat.*, 44-3:217–239, 2003.

[3] G. Birkhoff. On the groups of automorphisms. *Revista Unió Mat. Argentina*, 11:155–157, 1946. In Spanish.

[4] J. de Groot. Groups represented by homeomorphism groups. *Math. Ann.*, 138:80–102, 1959.

[5] M. Funk, O. H. Kegel, and K. Strambach. Gruppenuniversalität und homogenisierbarkeits. *Ann. Math. Pura Appl.*, 141:1–126, 1985.

[6] V. Koubek. Set functors. *Comment. Math. Univ. Carolinae*, 12:175–195, 1971.

[7] V. Koubek. Set functors II - contravariant case. *Comment. Math. Univ. Carolinae*, 14:47–59, 1973.

[8] V. Koubek and J. Reiterman. Set functors III - monomorphisms, epimorphisms, isomorphisms. *Comment. Math. Univ. Carolinae*, 14:441–455, 1973.

[9] A. Pultr and V. Trnková. *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*. North Holland and Academia, Praha, 1980.

[10] Y. T. Rhineghost. The functor that wouldn't be - a contribution to the theory of things that fail to exist. *in: Categorical Perspectives, Birkhauser-Verlag, Trends in Mathematics*, pages 29–36, 2001.

[11] Y. T. Rhineghost. The emergence of functors - a continuation of "the functor that wouldn't be". *in: Categorical Perspectives, Birkhauser-Verlag, Trends in Mathematics*, pages 37–46, 2001.

[12] V. Trnková. Some properties of set functors. *Comment. Math. Univ. Carolinae*, 10:323–352, 1969.

[13] V. Trnková. On descriptive classification of set functors I. *Comment. Math. Univ. Carolinae*, 12:143–175, 1971.

[14] V. Trnková. On descriptive classification of set functors II. *Comment. Math. Univ. Carolinae*, 12:345–357, 1971.

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