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GENERIC MORPHISMS, PARAMETRIC REPRESENTATIONS AND WEAKLY CARTESIAN MONADS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

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ABSTRACT. Two notions, generic morphisms and parametric representations, useful for the analysis of endofunctors arising in enumerative combinatorics, higher dimensional category theory, and logic, are defined and examined. Applications to the Batanin approach to higher category theory, Joyal species and operads are provided.

1. Introduction

Combinatorial species and analytic functors were introduced by André Joyal as a unifying conceptual notion for enumerative combinatorics. In characterising the analytic endofunctors of **Set** in [Joy86], Joyal was led to the technical notion of generic element of an endofunctor T of **Set**. Viewing the elements of T as functions $1 \to TX$, the property of genericness does not rely on the domain of these functions being the singleton, or on the category of sets. In this way, one arrives at the notion of a generic morphism $f : A \to TX$ for an endofunctor T of an arbitrary category. A stricter notion of generic morphism arose in the PhD thesis of Lamarche [Lam88], which itself builds on the work of Girard [Gir86] on qualitative domains. Qualitative domains give a semantics for variable types, and generic morphisms arise in two ways in this semantics. Namely, to express the normal form characterisation for stable maps of qualitative domains, and moreover, to express the normal form theorem for variable types (which arise as endofunctors of the category of qualitative domains). The paper [Gir86] inspired the development of a subject in logic called stable domain theory[1]. Applications to the study of stable domains and variable types, and their connections to higher category theory, are the subject of on-going research, but will not be discussed any further in this paper.

In his PhD thesis [Die77], Diers considered functors into **Set** with a family of representing objects, and in [CJ95] the theory of these familially representable functors was developed further. However, in higher dimensional category theory, there arise endofunctors of presheaf categories that are of a similar form. All of the formidable combinatorics in

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[1]A survey of this area can be found in [Tay90].

the operadic approaches comes packaged as such endofunctors. So it seems that overcoming these complications will require further techniques and concepts for the manipulation and investigation of such endofunctors. The inadequacy of our present state of knowledge of such endofunctors and monads is expressed most forcefully in appendix C of [Lei03]. Generic morphisms and related notions enable us to generalise the theory of familially representable functors, so as to include familially representable endofunctors of presheaf categories.

Abstract clubs and cartesian monads, together with operads over them, provide an excellent improvement and generalisation of the usual notion of non-symmetric operad. However, the corresponding notion for symmetric operads is yet to be identified. Generic morphisms can be used to recognise cartesian and weakly cartesian transformations, and so we are able to generalise cartesian monads thereby providing a candidate notion. It is then demonstrated that symmetric operads in **Set** are indeed captured by this generalisation.

The background required to understand the examples is presented in sections (2)–(4). Generics, parametric representations, related concepts, and associated results are presented in sections (5)–(7), which constitute the technical heart of the paper.

In section (5), T-generic morphisms are defined for an endofunctor T of an arbitrary category $\mathcal{A}$. We also consider the T for which any map $f : A \to TB$ factors appropriately through a generic (such functors are said to "admit generic factorisations"), and natural transformations that preserve and reflect generics in the appropriate fashion. One of the main technical themes of this paper is that these generic properties of endofunctors and natural transformations, correspond to pullback-preservation for endofunctors and cartesianness for natural transformations. One has (as we shall see) the more convenient "generic properties" of endofunctors and natural transformations on the one hand, versus the more commonly-used "cartesian properties" of endofunctors and natural transformations on the other. Just as one has pullbacks and weak pullbacks, there are strict and non-strict generic morphisms, and so the full correspondence alluded to here goes as follows:

- endofunctors that admit *strict* generic factorisations correspond to endofunctors that preserve wide pullbacks.
- endofunctors that admit generic factorisations correspond to endofunctors that preserve *weak* wide pullbacks.
- natural transformations that preserve and reflect *strict* generic morphisms correspond to cartesian natural transformations.
- natural transformations that preserve and reflect generic morphisms correspond to weakly cartesian natural transformations.

Most of section (5) is devoted to spelling out the parts of this correspondence that are valid in general, as well as exploring the interplay between the generic properties for endofunctors and those for natural transformations. The remainder of this section explains the relationship of these notions with the parametric right adjoints of [Str00], and their compatibility with the composition of endofunctors.

Section (6) provides two results which require some further hypotheses on $\mathcal{A}$ and T. In the previous section, it is shown that the generic properties of endofunctors imply the corresponding cartesian properties in general. The first result, Theorem(6.6), is the converse result that the preservation of wide pullbacks by an endofunctor implies that it admits strict generic factorisations. This result is not new – versions of it appear in [Die77], [Lam88], and [Tay88], although I know of no published proof. The second result, Theorem(6.8), is new, generalising André Joyal's observation that analytic endofunctors of **Set** preserve cofiltered limits.

The strict generic notions have an alternative 2-categorical description, in terms of the concept of a parametric representation described in section (7). Although this paper is concerned with *endo*functors, many of the notions, and parametric representations in particular, make sense for functors with different domain and codomain. Seen in this light, parametric representability is a very general notion of representability – including the familially representable functors of [CJ95] – but which on the face of it appears to have very little to do with the category **Set**. Instead such functors are between categories which have a small dense subcategory and satisfy a cocompleteness condition with respect to this subcategory. One sees the objects of this small dense subcategory as "element parametrizers", and these categories as being "accessible with respect to their elements". Such categories arise already in section (6) as part of the additional hypotheses required there. The remainder of section (7) relates parametric representability to strict generics and related notions, and generalises the characterisation of familial representability in [CJ95].

The applications are presented in sections (8)–(11). Section (8) characterises connected limit preserving endofunctors of presheaf categories with rank, and section (9) applies this to the description of some of the more fundamental combinatorial objects in the Batanin approach to higher category theory. Section (10) uses generics to characterise analytic endofunctors of **Set**, and section (11) describes our generalised notion of cartesian monad and operad over it.

2. Abstract Clubs

Throughout this paper we shall refer to a monad whose underlying endofunctor is T, unit is η and multiplication is μ by (T, η, μ) or just by T when the context is clear. We shall always use the letter η for the unit of a monad or an adjunction, the letter μ for the multiplication of a monad, and the letter ε for the counit of an adjunction. For a category $\mathcal{A}$ we write $\mathrm{End}(\mathcal{A})$ for the category of endofunctors of $\mathcal{A}$ and natural transformations between them. Recognising the strict monoidal structure on $\mathrm{End}(\mathcal{A})$ whose tensor product is composition of endofunctors, we write $\mathrm{Mnd}(\mathcal{A})$ for the category of monoids in $\mathrm{End}(\mathcal{A})$. The objects of this category are of course monads, and we shall refer to its arrows as monad morphisms. While there are other important notions of monad morphism, see [Str72] for example, they will not be discussed here.

2.1. Definition. *Let $\mathcal{A}$ be a category with finite limits, and $T \in \mathrm{End}(\mathcal{A})$. T-Coll is the full subcategory of $\mathrm{End}(\mathcal{A}){\downarrow}T$ consisting of the cartesian transformations into T. Write*

$$T\text{-Coll} \xrightarrow{\pi_T} \mathrm{End}(\mathcal{A})$$

for the faithful "projection", whose object map takes the domain of a cartesian transformation. When the context is clear, the T subscript for π will be dropped.

By the elementary properties of pullbacks, to give an object $S{\Rightarrow}T$ of T-Coll, it suffices to provide f, and for each $A \in \mathcal{A}$, the projections of

$$\begin{tikzcd} SA \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & TA \arrow[d, "T!"] \\ S1 \arrow[r, "f"'] & T1 \end{tikzcd}$$

More formally, the functor obtained by evaluating at the terminal object

$$T\text{-Coll} \xrightarrow{ev_1} \mathcal{A}{\downarrow}T1$$

is an equivalence of categories.

Given a monoid M in any monoidal category $\mathcal{V}$, the slice category $\mathcal{V}{\downarrow}M$ inherits a monoidal structure. The unit for $\mathcal{V}{\downarrow}M$ is the unit $i : I{\rightarrow}M$ for the monoid, and the tensor product of two objects f and g of $\mathcal{V}{\downarrow}M$ is the composite

$$A \otimes B \xrightarrow{f \otimes g} M \otimes M \xrightarrow{m} M$$

where m is the monoid multiplication. The projection

$$\mathcal{V}{\downarrow}M \longrightarrow \mathcal{V}$$

is a strict monoidal faithful functor. A monoid structure on $q : N{\rightarrow}M$ in $\mathcal{V}{\downarrow}M$, is a monoid structure on N in $\mathcal{V}$, for which q is a morphism of monoids. In particular when T is a monad on $\mathcal{A}$ then $\mathrm{End}(\mathcal{A}){\downarrow}T$ is a strict monoidal category, since composition of endofunctors makes $\mathrm{End}(\mathcal{A})$ strict monoidal. A subcategory $\mathcal{W}$ of a monoidal category $\mathcal{V}$ is said to be a *monoidal subcategory* of $\mathcal{V}$ when $\mathcal{W}$ is a monoidal category and the inclusion of $\mathcal{W}$ in $\mathcal{V}$ is a strict monoidal functor.

2.2. Definition. [Kelly [Kel92]] *A* club *on $\mathcal{A}$ is a $T \in \mathrm{Mnd}(\mathcal{A})$ such that T-Coll is a monoidal subcategory of $\mathrm{End}(\mathcal{A}){\downarrow}T$.*

We say that T *preserves cartesian transformations* when $\phi : S{\Rightarrow}S'$ in $\mathrm{End}(\mathcal{A})$ is a cartesian transformation implies $T\phi$ is cartesian also. A more explicit description of clubs is given by

2.3. Proposition. [Kelly [Kel92]] *A monad T is a club iff η and μ are cartesian, and T preserves cartesian transformations $\phi : S \Rightarrow T$ (that is, $T\phi$ is cartesian for cartesian transformations ϕ whose codomain is T).*

It is worth describing the monoidal structure on T-Coll from the point of view of $\mathcal{A}{\downarrow}T1$. The unit is simply the terminal component of the unit of T. Given $f, g \in \mathcal{A}{\downarrow}T1$, their tensor product is the dotted composite in

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl] \arrow[dr, dotted] & & & \\ A \arrow[dr, "f"'] & \text{pb} & TB \arrow[dl, "T!"] \arrow[dr, dotted, "Tg"] & & \\ & T1 & & T^2 1 \arrow[dr, dotted, "\mu_1"] & \\ & & & & T1 \end{tikzcd}$$

2.4. Definition. *Let T be a club. The category* T-Op *of* T-operads *is the category of monoids in* T-Coll. *Explicitly, a T-operad is a cartesian monad morphism into T, and a morphism $\phi \to \phi'$ in* T-Op *is a commutative triangle*

$$\begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & S' \arrow[dl, "\phi'"] \\ & T & \end{tikzcd}$$

in $\mathrm{Mnd}(\mathcal{A})$. *By the elementary properties of pullbacks, ψ here is automatically a cartesian monad morphism.*

A *cartesian monad* is a club whose functor part preserves pullbacks. That is, a monad T on a finitely complete category $\mathcal{A}$ for which T preserves pullbacks and μ and η are cartesian. A T-operad in our sense, is precisely a T-multicategory in the sense of Burroni [Bur71], Hermida [Her00b] and Leinster [Lei00], whose underlying object is the terminal object of $\mathcal{A}$.

2.5. Definition. *The category of algebras for an operad $\phi : S \Rightarrow T$ is the category of Eilenberg-Moore algebras for the monad S.*

2.6. Proposition. [Kelly [Kel92]] *Let T be a club and $\phi : S \Rightarrow T$ be an operad. Then S is a club.*

2.7. Examples.

1. Recall the monad $\mathcal{M}$ on **Set** whose algebras are monoids. Let X be a set. Then $\mathcal{M}X$, often denoted as X^*, is the set of finite sequences from X. We shall denote a typical element of $\mathcal{M}X$ as a function $n \to X$ where $n \in \mathbb{N}$ is also being regarded as

the set $\{0, ..., n-1\}$. The component of the unit of $\mathcal{M}$ at a set X takes $x \in X$ to the function $x : 1 \to X$ which picks out the element x. An element of $\mathcal{MM}X$ is a finite sequence of finite sequences from X which is more conveniently regarded as

$$k \xleftarrow{f} n \xrightarrow{x} X$$

where f is in $\mathbf{\Delta}$ (the category of finite ordinals and order-preserving functions), and x is just a function. So, as far as f is concerned, the set n is being regarded as an ordinal in the usual way, whereas from the point of view of x, n is just a set. The monad multiplication applied to this element forgets f. $\mathcal{M}$ preserves connected limits, and its unit and multiplication are cartesian[2]. An $\mathcal{M}$-operad is a non-symmetric operad in **Set**.

2. The tree monad $\mathcal{T}$ on **Glob** (the category of globular sets) described in section (9) is cartesian and, as pointed out in [Lei00], a $\mathcal{T}$-operad is an operad in the monoidal globular category **Span**(**Set**) in the sense of [Bat98]. Among these operads is one whose algebras are weak ω-categories in the sense of Michael Batanin.

3. In [Web01] a cartesian monad $\mathcal{F}$ on **Glob** is described whose algebras are strict monoidal strict ω-categories. There is an $\mathcal{F}$-operad whose algebras are monoidal weak ω-categories. These are one-object Batanin weak ω-categories whose n-cells for $n > 0$ are being regarded as $(n-1)$-cells.

4. Recall the monad $\mathcal{S}$ on **Cat** whose algebras are symmetric strict monoidal categories. The category $\mathcal{S}X$ has the following description

   - objects: functors $n \to X$ where n is being regarded as a discrete category.
   - arrows: an arrow from x to y consists of a pair (f, ϕ) as in

$$\begin{tikzcd} n \arrow[rr, "\phi"] \arrow[dr, "x"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & n \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

   where $\phi \in \mathrm{Sym}_n$ and f is a natural transformation as shown.

This monad is also cartesian, and $\mathcal{S}$-operads are the clubs originally considered by Max Kelly in [Kel72b] and [Kel72a]. Monoidal categories, strict monoidal categories, braided monoidal categories, braided strict monoidal categories, symmetric monoidal categories, and symmetric strict monoidal categories are examples of categorical structures that arise as algebras of $\mathcal{S}$-clubs.

[2]In fact it is shown in [Bén91] that this is true far more generally, that is, when **Set** is replaced by an elementary topos with a natural numbers object.

5. Recall the monad $\mathcal{C}$ on **Set** whose algebras are commutative monoids. An element of $\mathcal{C}X$, the free commutative monoid on the set X, is an unordered sequence from X. This amounts to an equivalence class of elements of $\mathcal{M}(X)$ in which x and x' are considered equivalent iff there is a $\rho \in \mathrm{Sym}_n$ (the n^{th} symmetric group) making

$$\begin{tikzcd} n \arrow[rr, "\rho"] \arrow[dr, "x"'] & & n \arrow[dl, "x'"] \\ & X & \end{tikzcd}$$

commute. We shall use square brackets to denote the taking of such equivalence classes, so such an element will be denoted by $[x]$ and x is called a *representative* of this element of $\mathcal{C}(X)$. Similarly an element of $\mathcal{C}\mathcal{C}X$ is represented by an element of $\mathcal{M}\mathcal{M}X$, regarded as in (2.7)(1) modulo the identification of the rows of

$$\begin{tikzcd} k \arrow[d, "\rho_1"'] & n \arrow[l] \arrow[r] \arrow[d, "\rho_0"'] & X \arrow[d, equal] \\ k & n \arrow[l] \arrow[r] & X \end{tikzcd}$$

given the existence of permutations ρ_i for $i \in 2$, making the above diagram commute. The rest of this monad is specified as in (2.7)(1), that is, on representatives. That is, the taking of equivalence classes is a monad morphism $c : \mathcal{M} \Rightarrow \mathcal{C}$. $\mathcal{C}$ is not cartesian, because it does not preserve the pullback

$$\begin{tikzcd} 2 \times 2 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 2 \arrow[r] & 1 \end{tikzcd}$$

and because the naturality square of μ for $2 \to 1$ is not cartesian. However, in section (11) the current notion of club and operad for it will be generalised to include $\mathcal{C}$. Given this definition, $\mathcal{C}$-operads coincide with symmetric operads in **Set**.

Many more examples are presented in [Lei03].

3. Analytic endofunctors of **Set** I

We shall denote by G the composite

$$\mathbf{Set}/\mathbb{N} \xrightarrow{\mathrm{ev}'_1} \mathcal{M}\text{-Coll} \xrightarrow{\pi_{\mathcal{M}}} \mathrm{End}(\mathbf{Set})$$

where ev'_1 is a pseudo-inverse of ev_1. Upon identifying $\mathbf{Set}/\mathbb{N}$ as $[\mathbb{N}, \mathbf{Set}]$, G can also be regarded as the process of taking left extensions along the functor $E_{\mathcal{M}} : \mathbb{N} \to \mathbf{Set}$, which regards $n \in \mathbb{N}$ as in (1). Clearly $E_{\mathcal{M}}$ factors through $\mathbf{Set}_f$, and so $G(\alpha : A \to \mathbb{N})$ is finitary.

An element of $G(\alpha)(X)$ is a pair of functions $(1 \to A_n, n \to X)$ where A_n is the fibre of α over n. A morphism $f : X_1 \to X_2$ in $\mathcal{X}$ is said to be *essentially in the image* of a functor $F : \mathcal{A} \to \mathcal{X}$ when there is an $g : A_1 \to A_2$ in $\mathcal{A}$ and isomorphisms $\rho_i : FA_i \to X_i$ such that $\rho_2 Fg = f\rho_1$. An object X in $\mathcal{X}$ is essentially in the image of F when 1_X is essentially in the image of F.

3.1. Definition. *The functors and natural transformations essentially in the image of G are said to be* strongly analytic.

Define $\mathbb{P}$ to be the category with natural numbers as objects, hom-sets given by

$$\mathbb{P}(n,m) = \begin{cases} \mathrm{Sym}_n & \text{if } n = m \\ \emptyset & \text{otherwise} \end{cases}$$

and composition given by multiplication in the groups Sym_n[3]. A *species* is a functor $\mathbb{P} \to \mathbf{Set}$. Recall the adjunction

$$\begin{tikzcd} \mathrm{End}(\mathbf{Set}) \arrow[r, bend right, "r"'] \arrow[r, phantom, "\perp"] & {[\mathbb{P}, \mathbf{Set}]} \arrow[l, bend right, "E"'] \end{tikzcd}$$

which corresponds to left extension and restriction along the $E_{\mathcal{C}} : \mathbb{P} \to \mathbf{Set}$ that regards permutations as bijective functions. Clearly $E_{\mathcal{C}}$ factors through $\mathbf{Set}_f$, and so EX is finitary for any species X. Given a set Z, an element of $E(X)(Z)$ is represented by a pair $(x : 1 \to X_n, h : n \to Z)$ modulo the identification of (x,h) with (x',h') whenever there is a $\rho \in \mathrm{Sym}_n$ such that the triangles

$$\begin{tikzcd} & 1 \arrow[dl, "x"'] \arrow[dr, "x'"] & \\ X_n \arrow[rr, "X_\rho"'] & & X_n \end{tikzcd} \qquad \begin{tikzcd} n \arrow[rr, "\rho"] \arrow[dr, "h"'] & & n \arrow[dl, "h'"] \\ & Z & \end{tikzcd}$$

commute. Denote by $[x,h]$ the element of $E(X)(Z)$ represented by (x,h). Using this notation, the arrow map of $E(X)$ is simply

$$[x,h] \xmapsto{E(X)(f)} [x, fh]$$

3.2. Definition. *The functors and natural transformations essentially in the image of E are said to be* analytic.

In [Joy86] the following theorem was obtained.

[3] More conceptually, $\mathbb{P} \cong \mathcal{S}1$.

3.3. Theorem. [Joyal]

1. *An endofunctor of* **Set** *is analytic iff it preserves weak pullbacks, filtered colimits and cofiltered limits.*

2. *A natural transformation between analytic endofunctors of* **Set** *is analytic iff it's naturality squares are weak pullbacks.*

It was close inspection of Joyal's proof of this result that lead to much of the theory described in this paper. One immediate consequence of this result is that analytic functors compose, a fact that this not obvious from the definition.

The unit η and counit ε of $E \dashv r$ will now be specified.

- For $X \in [\mathbb{P}, \mathbf{Set}]$ and $x \in X_n$, $\eta_{X,n}(x) = [x, 1_n]$.

- For $T \in \mathrm{End}(\mathbf{Set})$ and $Z \in \mathbf{Set}$, $\varepsilon_{T,Z}[a : 1 \to T(n), h : n \to Z] = Th(a)$.

Identifying $\mathbf{Set}/\mathbb{N}$ as $[\mathbb{N}, \mathbf{Set}]$, the adjunction $F \dashv R$ arises by left kan extension and restriction along the identity on objects functor $\mathbb{N} \to \mathbb{P}$, and we have

$$\begin{tikzcd}
\mathbf{Set}/\mathbb{N} \arrow[dd, "F"] \arrow[dr, "G"] & \\
& \mathrm{End}(\mathbf{Set}) \\
{[\mathbb{P}, \mathbf{Set}]} \arrow[uu, bend left, "R"] \arrow[ur, "E"'] &
\end{tikzcd}$$

with $EF = G$. Let X be a species and write ε for the counit of the above adjunction, then for $Z \in \mathbf{Set}$, $(E\varepsilon_X)_Z(x, h) = [x, h]$. For T analytic, denote the corresponding natural transformation as

$$\overline{T} \xrightarrow{c_T} T$$

That is, for $T = EX$, $c_T = E\varepsilon_X$. Observe that $\overline{T}$ is strongly analytic by definition, and that from the explicit description of c_T, its components are clearly surjective.

3.4. Example. For $T = \mathcal{C}$, c_T is (the underlying natural transformation of) the monad morphism c described in (5).

4. Globular cardinals

Define the category $\mathbb{G}$ to have natural numbers as objects, and a generating subgraph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} 3 \underset{\tau_3}{\overset{\sigma_3}{\rightrightarrows}} \cdots$$

subject to the "cosource/cotarget" equations $\sigma_{n+1}\sigma_n = \tau_{n+1}\sigma_n$ and $\tau_{n+1}\tau_n = \sigma_{n+1}\tau_n$, for every $n \in \mathbb{N}$. More generally, an arrow $n \longrightarrow m$ in $\mathbb{G}$ is a string of σ's and τ's of length $m-n$, and so by the cosource/cotarget equations, is determined by the first (ie right-most) character. So when $n < m$ we can write

$$\begin{tikzcd} n \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & m \end{tikzcd}$$

to describe the hom-set $\mathbb{G}(n, m)$. The category **Glob** of *globular sets* is defined as $\mathbf{Glob} := [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$. Thus, a globular set Z consists of a diagram of sets and functions

$$\begin{tikzcd} Z_0 & Z_1 \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & Z_2 \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] & Z_3 \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "t_2"] & \cdots \arrow[l, shift right, "s_3"'] \arrow[l, shift left, "t_3"] \end{tikzcd}$$

so that $s_n s_{n+1} = s_n t_{n+1}$ and $t_n t_{n+1} = t_n s_{n+1}$ for every $n \in \mathbb{N}$. The elements of Z_n are called the n-cells of Z, and the functions s_n and t_n are called source and target functions. In this way an $(n+1)$-cell z is regarded as a directed edge $s(z) \xrightarrow{z} t(z)$ between n-cells. The source/target equations tell us how to regard an $(n+2)$-cell z as an edge between edges between n-cells. Clearly, every n-cell has a unique k-source and a unique k-target, where $0 \leq k < n$.

We shall now describe globular pasting schemes. Let Z be a globular set. Recall from [Str91] the *solid triangle order* $\blacktriangleleft$ on the elements (of all dimensions) of Z. Define first the relation $x \prec y$ for $x \in Z_n$ iff $x = s_n(y)$ or $t_{n-1}(x) = y$. Then take $\blacktriangleleft$ to be the reflexive-transitive closure of $\prec$. Write $\mathrm{Sol}(Z)$ for the preordered set so obtained. Observe that Sol is the object map of a functor

$$\mathbf{Glob} \xrightarrow{\mathrm{Sol}} \mathbf{PreOrd}$$

where **PreOrd** is the category of preordered sets and order-preserving functions.

4.1. Definition. *A* globular cardinal *is a globular set Z such that* $\mathrm{Sol}(Z)$ *is a non-empty finite linear order.*

We begin understanding morphisms between globular cardinals by noticing that the square

$$\begin{tikzcd} \mathbf{Glob} \arrow[r, "\mathrm{Sol}"] \arrow[d, "\mathrm{res}"'] & \mathbf{PreOrd} \arrow[d, "U"] \\ \mathbf{Set}^{\mathbb{N}} \arrow[r, "\mathrm{colim}"'] & \mathbf{Set} \end{tikzcd}$$

is commutative up to isomorphism in **Cat**, where U is the forgetful functor and res is restriction along the inclusion of objects $\mathbb{N} \to \mathbb{G}$. Since U creates all limits and colimits, res preserves all limits and colimits, and coproducts in **Set** commute with colimits and connected limits, Sol preserves colimits and connected limits. Furthermore, observe that Sol preserves and reflects monics and epics since U, res and colim do. We summarise these observations in

4.2. Proposition. *The functor* Sol *preserves colimits and connected limits, and reflects monos and epis.*

4.3. Corollary.

1. *Let* $X \xrightarrow{f} Z \in \mathbf{Glob}$ *and* X, Z *be globular cardinals. Then* f *is monic.*

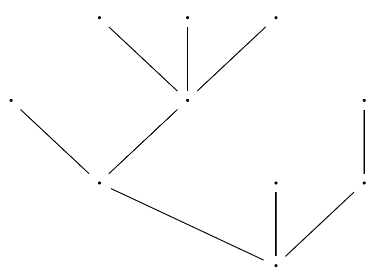
2. *Let* $X \xrightarrow{r} Z$ *be a retraction and* X *be a globular cardinal. Then* r *is an isomorphism.*

Proof. (1): Consecutive elements $x, y \in \mathrm{Sol}(X)$ come from different dimensions, so that $\mathrm{Sol}(f)(x) \neq \mathrm{Sol}(f)(y)$. Thus $\mathrm{Sol}(f)$ is an order-preserving function between finite linear orders that does not identify consecutive elements, and so must be monic. By (4.2) f must be monic.
(2): A retract of a finite linear order in **PreOrd** is a finite linear order. Thus Z is a globular cardinal and the result follows from (1). ■

It is a result of Street [Str00] that globular cardinals are the globular pasting schemes of [Bat98]. We shall describe that connection now for the convenience of the reader. A *tree*[4] T of height n is a diagram

$$T_0 \xleftarrow{\partial_0} \cdots \xleftarrow{\partial_{n-1}} T_n$$

in $\mathbf{\Delta}$ such that $T_0 = 1$. A *leaf* of height k ($0 \leq k < n$) is an element of T_k which is not in the image of ∂_k. All elements of T_n are leaves (of height n). For example



is a tree T of height 3, for which T_1, T_2 and T_3 each have 3 elements. It has 2 leaves of height 2 and 1 leaf of height 1. For $r \in \mathbb{N}$, define an *r-zig-zag* sequence to be a finite sequence of natural numbers

$$(n_i : i \in (2r-1))$$

such that $n_{2j} > n_{2j+1} < n_{2j+2}$ for $j \in (r-1)$. Given a tree T with r leaves, we construct an r-zig-zag sequence, called the *zig-zag sequence of* T,

$$\mathrm{zz}(T) := (n_{T,i} : i \in (2r-1))$$

[4] Usually a tree is defined to be a finite undirected loop-free graph. The trees described here are rooted trees, that is, trees together with a distinguished vertex (the root which is the element of T_0).

by ordering the leaves in the obvious way (from left to right in the above example), taking $n_{T,2j}$ to be the height of the j-th leaf (where $j \in r$), and taking $n_{T,2j+1}$ to be the maximum height at which the j-th and $(j+1)$-st leaves are joined (where $j \in (r-1)$). By induction on r, this construction specifies a bijection between trees with r leaves, and r-zig-zag sequences. A *smooth zig-zag sequence* is a finite sequence

$$(n_i : i \in r)$$

of natural numbers, satisfying

1. $n_0 = 0 = n_{r-1}$.
2. For $i \in (r-1)$, $|n_{i+1} - n_i| = 1$.

A *peak* of a smooth zig-zag sequence is an n_i, where $0 < i < (r-1)$, such that $n_{i-1} < n_i > n_{i+1}$. A *trough* of a smooth zig-zag sequence is an n_i, where $0 < i < (r-1)$, such that $n_{i-1} > n_i < n_{i+1}$. Reading off the peaks and troughs of smooth zig-zag sequences as they arise, provides a bijection between smooth zig-zag sequences and zig-zag sequences, and thus with trees. For any tree T, define $\mathrm{szz}(T)$, the *smooth zig zag sequence of* T, to be the smooth zig-zag sequence corresponding to T by these bijections. In the above example

$$\mathrm{zz}(T) = (2, 1, 3, 2, 3, 2, 3, 0, 1, 0, 2)$$

and

$$\mathrm{szz}(T) = (0, 1, 2, 1, 2, 3, 2, 3, 2, 3, 2, 1, 0, 1, 0, 1, 2, 1, 0)$$

To obtain the tree T corresponding to a globular cardinal Z, write down the dimensions of the elements of Z as they appear in $\mathrm{Sol}(Z)$. The result is a smooth zig-zag sequence, and the corresponding tree is T. Conversely, given a tree T, we regard its zig-zag sequence as a diagram of representables in **Glob**

$$n_{T,0} \xleftarrow{\tau} n_{T,1} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} n_{T,2r}$$

identifying $\mathbb{G}$ as a full subcategory of **Glob** in the usual way. Then, the globular set associated to T is the colimit of this diagram. This construction of globular sets from trees was named the **-construction* in [Bat98].

Identifying $\mathbf{\Delta}$ as a subcategory of **Set** as usual, the full subcategory **tr** of **Glob** consisting of the globular cardinals, is clearly isomorphic to the subcategory of $\mathbf{\Delta}{\downarrow}\mathbb{N}$ consisting of

- objects: smooth zig-zag sequences. Here a sequence of natural numbers of length r is being regarded as a function $r \longrightarrow \mathbb{N}$.
- arrows: injections which preserve sources and targets. Recall that the source(target) of an instance of n in a smooth zig-zag sequence, is the preceding(succeeding) instance of $(n-1)$.

Trees form a globular set **Tr**. The set of n-cells consists of trees of height $\leq n$. Let T be a tree of height n and $0 \leq m < n$. Then the *truncation* at height m of T, notated by $\partial_m(T)$, is obtained by ignoring the vertices of T above height m. Sources and targets for **Tr** are given by $s_m = \partial_m = t_m$. There is a morphism $u : 1 \to \mathbf{Tr}$ in **Glob** which in each dimension n, picks out the tree U_n, which has one leaf at height n. That is, U_n is the tree whose corresponding globular set is the representable $\mathbb{G}(-, n)$.

Truncation admits a natural description via smooth zig-zag sequences and the above description of **tr**. Let $X = (n_i : i \in r)$ be a smooth zig-zag sequence. An *m-region* of X is a subsequence Y such that

1. Y is *consecutive*, that is, if $0 \leq i < j < k < r$, and $n_i, n_k \in Y$, then $n_j \in Y$.
2. the first and last terms of Y are instances of m.
3. for any $n_i \in Y$, $n_i \geq m$.
4. Y is a maximal subsequence of X for which conditions (1)-(3) hold.

Let T be a tree of height n and $m \in \mathbb{N}$. Then one obtains $\mathrm{szz}(\partial_m(T))$ by collapsing each of the m-regions of $\mathrm{szz}(T)$ to a single instance of m. There is a morphism $\sigma : \partial_m(T) \to T$ in **Glob**, which identifies the $n_i < m$, and takes each instance of $m \in \mathrm{szz}(\partial_m(T))$ to the first term in the corresponding m-region of $\mathrm{szz}(T)$. Similarly there is a morphism $\sigma : \partial_m(T) \to T$ in **Glob**, which identifies the $n_i < m$, and takes each instance of $m \in \mathrm{szz}(\partial_m(T))$ to the last term in the corresponding m-region of $\mathrm{szz}(T)$. Notice in particular when $n \leq m$, that σ and τ are identities. The following proposition is immediate.

4.4. Proposition. *Let $T \in \mathbf{Tr}_n$, $0 \leq r < m < n$ and consider*

$$\begin{tikzcd} \partial_r(T) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \partial_m(T) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & T \end{tikzcd}$$

Then $\sigma\sigma = \sigma = \tau\sigma$ and $\tau\tau = \tau = \sigma\tau$.

For example, restricting to the trees with one leaf, whose corresponding globular sets are the representables, one recaptures the cosource-cotarget relations. Proposition(4.4) enables one to write the *-construction as a functor in the following definition.

4.5. Definition. *The functor $E_{\mathcal{T}} : y{\downarrow}\mathbf{Tr} \to \mathbf{Glob}$, where y is the yoneda embedding, is defined to have the arrow map*

$$\begin{tikzcd} (\partial_m(T), m) \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & & \partial_m(T) \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\ (T, n) \arrow[r, phantom, "\longmapsto"] & & T \end{tikzcd}$$

where $m \leq n$.

Suppose that S and T are trees, and $m \in \mathbb{N}$, such that $\partial_m(S) = \partial_m(T)$. We shall construct a new tree $S \otimes_m T$ so that $\partial_m(S \otimes_m T) = \partial_m(S) = \partial_m(T)$. Notice that this condition ensures that S, T and $S \otimes_m T$ must have the same number of m-regions. It suffices to specify the m-regions of $S \otimes_m T$. The i-th m-region of $S \otimes_m T$ is obtained by concatenating the i-th m-region S_i, of S, with the i-th m-region T_i of T, and identifying the last term of S_i with the first term of T_i (which are both instances of m).

4.6. Example. Let $\mathrm{szz}(S)$ be

$$(0, 1, 2, 1, 0, 1, 2, 3, 2, 1, 2, 3, 2, 3, 2, 1, 0)$$

and $\mathrm{szz}(T)$ be

$$(0, 1, 2, 3, 4, 3, 2, 1, 0, 1, 2, 3, 2, 1, 2, 1, 0)$$

then $\mathrm{szz}(\partial_2(S)) = \mathrm{szz}(\partial_2(T))$ is

$$(0, 1, 2, 1, 0, 1, 2, 1, 2, 1, 0)$$

The three 2-regions of S as they arise are (2), $(2, 3, 2)$ and $(2, 3, 2, 3, 2)$. The corresponding three 2-regions of T are $(2, 3, 4, 3, 2)$, $(2, 3, 2)$ and (2) respectively. Thus, the corresponding three 2-regions of $S \otimes_2 T$ are $(2, 3, 4, 3, 2)$, $(2, 3, 2, 3, 2)$ and $(2, 3, 2, 3, 2)$ respectively. Thus, $\mathrm{szz}(S \otimes_2 T)$ is

$$(0, 1, 2, 3, 4, 3, 2, 1, 0, 1, 2, 3, 2, 3, 2, 1, 2, 3, 2, 3, 2, 1, 0)$$

It is instructive to picture this example in terms of trees.

Writing

$$S \xrightarrow{c_S} S \otimes_m T \xleftarrow{c_T} T$$

for the obvious inclusions, it is straightforward to observe that

$$\begin{tikzcd} \partial_m(S) \arrow[r, "\sigma"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\mathrm{po}"] & T \arrow[d, "c_T"] \\ S \arrow[r, "c_S"'] & S \otimes_m T \end{tikzcd} \tag{1}$$

is a pushout in **Glob**. Let $n \in \mathbb{N}$. When $n > m$, n-regions are contained within m-regions, so that n-truncation and $\otimes_m$ "commute". That is, $\partial_n(S \otimes_m T) = \partial_n(S) \otimes_m \partial_n(T)$. Conversely, when $n \leq m$, $\partial_n \partial_m = \partial_n$, and $\partial_n(S)$ and $\partial_n(T)$ have no m-regions, so that $\partial_n(S \otimes_m T) = \partial_n(S) \otimes_m \partial_n(T)$ holds in this case also. It is also straightforward to see that truncation commutes with cosources and cotargets of globular cardinals.

Let A be a globular cardinal whose corresponding zig-zag sequence is $(n_{A,i} : i \in (2r-1))$. A morphism of globular sets $f : A \rightarrow \mathbf{Tr}$ amounts to a sequence $(T_i : i \in r)$ of trees, where T_i is of height $\leq n_{A,2i}$, and for $i \in (r-1)$, $\partial_{n_{T,2i+1}}(T_i) = \partial_{n_{T,2i+1}}(T_{i+1})$. The comments relating to $\otimes_m$ generalise to

4.7. Proposition. *Let A be a globular cardinal, $f : A \to \mathbf{Tr}$, and*

$$\begin{tikzcd} y{\downarrow}A \arrow[r] \arrow[d, "f\circ -"'] \arrow[dr, phantom, "\overset{g}{\Longrightarrow}"] & 1 \arrow[d, "B"] \\ y{\downarrow}\mathbf{Tr} \arrow[r, "E_{\mathcal{T}}"'] & \mathbf{Glob} \end{tikzcd}$$

exhibits B as a left extension. That is, g is the universal cocone that exhibits B as $\mathrm{colim}(E_{\mathcal{T}}(f \circ -))$*. Then B is a globular cardinal. Moreover, such colimits commute with cosources and cotargets of globular cardinals.*

Proof. B is the colimit of the diagram

$$T_0 \xleftarrow{\tau} \partial_{n_{T,1}}(T_0) \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} T_{r-1}$$

where $(T_i : i \in r)$ is the sequence of trees corresponding to f, and these trees are being regarded as globular cardinals. This colimit can be obtained by successively taking pushouts of the form (1). As argued above, such pushouts give rise to globular cardinals, and commute with cosources and cotargets. ∎

Let A and B be trees of height n. A morphism $f : B \to A$ of trees is a commutative diagram

$$\begin{tikzcd} B_0 \arrow[d, "f_0"'] & B_1 \arrow[l, "\partial_0"'] \arrow[d, "f_1"'] & \cdots \arrow[l, "\partial_1"'] & B_n \arrow[l, "\partial_{n-1}"'] \arrow[d, "f_n"'] \\ A_0 & A_1 \arrow[l, "\partial_0"] & \cdots \arrow[l, "\partial_1"] & A_n \arrow[l, "\partial_{n-1}"] \end{tikzcd}$$

in $\mathbf{Set}$ such that for $i \in n$, f_{i+1} preserves the linear order on each fibre of ∂_i. Let A have r leaves. Regard j-th leaf of A (ie $j \in r$), say of height $h(j)$, as being picked out by a morphism of trees $l(j) : U_{h(j)} \to A$. Pulling back the $l(j)_i$ along the f_i in $\mathbf{Set}$ distinguishes a subtree $f^{-1}(j)$ of B. Doing this for each leaf of A produces $\hat{f} : A \to \mathbf{Tr}$ in $\mathbf{Glob}$ (where A here is regarded as a globular cardinal). That is, $\hat{f}$ corresponds to the sequence of trees $(f^{-1}(j) : j \in r)$. Moreover $\mathrm{colim}(E_{\mathcal{T}}(\hat{f} \circ -)) \cong A$ and the universal cocone is uniquely determined. We have proved

4.8. Proposition. *The above construction sets up a bijective correspondence between morphisms of plane trees $f : B \to A$, and $\hat{f} : A \to \mathbf{Tr}$ and*

$$\begin{tikzcd} y{\downarrow}A \arrow[r] \arrow[d, "\hat{f}\circ -"'] \arrow[dr, phantom, "\overset{g}{\Longrightarrow}"] & 1 \arrow[d, "B"] \\ y{\downarrow}\mathbf{Tr} \arrow[r, "E_{\mathcal{T}}"'] & \mathbf{Glob} \end{tikzcd}$$

exhibiting B as a left extension.

We write Ω for the category whose objects are globular cardinals and morphisms are morphisms between the corresponding trees[5].

5. Generic morphisms

Throughout this section, take $\mathcal{A}$ to be a category, S and T to be endofunctors of $\mathcal{A}$, and $\phi : S \Rightarrow T$ to be a natural transformation from S to T.

5.1. Definition.

1. *Let I be a non-empty set. A diagram in $\mathcal{A}$ consisting of the family of arrows*

$$((\pi_i : W \to B_i, f_i : B_i \to C) : i \in I)$$

such that $f_i\pi_i = f_j\pi_j$ for every $i, j \in I$ is called a wide commutative square. *It will be convenient to denote its common diagonal $f_i\pi_i$ by π, to drop the adjective "commutative" when the context is clear, and to drop the adjective "wide" when the cardinality of I is 2. Such a wide square is said to be* weakly cartesian relative to $A \in \mathcal{A}$ *if for any other wide commutative square*

$$((a_i : A \to B_i, f_i : B_i \to C) : i \in I)$$

there is a $z : A \to W$ such that $a_i = \pi_i z$ for every $i \in I$. When these commutative fillers (ie the z's) exist uniquely, the original wide square is said to be cartesian relative to A. *This wide square is said to be* weakly cartesian *(also a* wide weak pullback*), respectively* cartesian *or a* wide pullback, *when it is weakly cartesian, respectively cartesian, relative to all $A \in \mathcal{A}$.*

2. *ϕ is (weakly) cartesian (relative to A) when its naturality squares are (weakly) cartesian (relative to A).*

5.2. Definition. *A morphism $f : A \to TB$ is* T-generic *when for any α, β, and γ making the outside of*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & T(X) \arrow[d, "T(\gamma)"] \\ T(B) \arrow[r, "T(\beta)"'] \arrow[ur, dotted, "T(\delta)"] & T(Y) \end{tikzcd}$$

*commute, there is a δ for which $\gamma \circ \delta = \beta$ and $T(\delta) \circ f = \alpha$. We call such a δ a T-*fill *for this square. We say that f is a* strict T-generic, *when there is a unique T-fill for any α, β, and γ as above.*

[5] Another important viewpoint described in [BS00] and [Her00a] is to regard Ω as a monoidal globular category which plays the role of a globular monoid classifier.

5.3. Example. Let T be an endofunctor of **Set**. A generic element of T in the sense of [Joy86] is a generic morphism $1 \to TB$ in our sense.

5.4. Definition. *T* admits (strict) generic factorisations relative to *$A \in \mathcal{A}$ when any $A \to TZ$ factors as*

$$A \xrightarrow{g} T(H) \xrightarrow{T(h)} T(Z)$$

where g is (strict) T-generic. We say that T admits (strict) generic factorisations *when it admits (strict) generic factorisations relative to all $A \in \mathcal{A}$.*

5.5. Example. Recall the monoid monad on **Set** from (2.7)(1). A function $f : A \to \mathcal{M}B$ amounts to functions $\overline{f}_a : n_a \to B$ where $n_a \in \mathbb{N}$ and $a \in A$. That is, f amounts to a discrete A-indexed cocone with vertex B. One can verify directly that f is $\mathcal{M}$-generic iff $\overline{f}$ is a universal (that is, a coproduct) cocone, and that $\mathcal{M}$ admits strict generic factorisations, or see this as a consequence of (7.3) below. Moreover $f : A \to \mathcal{M}B$ is generic and A is finite $\Rightarrow$ B is finite.

5.6. Definition.

1. *ϕ* preserves (strict) generics relative to *$A \in \mathcal{A}$ when for all B and f (strict) S-generic, the composite*

   $$A \xrightarrow{f} S(B) \xrightarrow{\phi_B} T(B)$$

   is a (strict) T-generic. ϕ preserves (strict) generics *when it preserves (strict) generics relative to all $A \in \mathcal{A}$.*

2. *ϕ* reflects (strict) generics relative to *$A \in \mathcal{A}$ when for all B and f, the composite*

   $$A \xrightarrow{f} S(B) \xrightarrow{\phi_B} T(B)$$

   is (strict) T-generic implies that f is (strict) S-generic. ϕ reflects (strict) generics *when it reflects (strict) generics relative to all $A \in \mathcal{A}$.*

We shall now describe the basic general facts concerning generic morphisms, endofunctors that admit generic factorisations and natural transformations that are (weakly) cartesian.

5.7. Lemma. *Let*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & T(C) \arrow[d, "T(k)"] \\ T(B) \arrow[r, "T(h)"'] & T(D) \end{tikzcd}$$

commute where f and g are T-generic. Then any T-fill $B \longrightarrow C$ for this square is an isomorphism.

Proof. Use the genericness of f to induce δ_1 and that of g to induce δ_2.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & T(C) \arrow[d, "T(k)"] \\ T(B) \arrow[r, "T(h)"'] \arrow[ur, dotted, "T(\delta_1)"] & T(D) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & T(B) \arrow[d, "T(\delta_1)"] \\ T(C) \arrow[r, "1"'] \arrow[ur, dotted, "T(\delta_2)"] & T(C) \end{tikzcd}$$

Thus δ_2 is a section of δ_1. By the same argument δ_2 is also a retraction, and so is the inverse of δ_1. ∎

5.8. Corollary. *T admits strict generic factorisations relative to A $\Rightarrow$ all generics $f : A \to T(B)$ are strict.*

Let $\mathcal{A}$ have a terminal object 1 and denote by $\hat{T} : \mathcal{A} \to \mathcal{A} \downarrow T1$ the functor which sends A to $T(!) : TA \to T1$. In [Str00] T is said to be a *parametric right adjoint* when $\hat{T}$ has a left adjoint.

5.9. Proposition. *Let $\mathcal{A}$ have a terminal object. T admits strict generic factorisations iff T is a parametric right adjoint.*

Proof. ($\Rightarrow$): Choose a strict generic factorisation

$$A \xrightarrow{g_f} TA_f \xrightarrow{T!} T1$$

for every morphism $f : A \to T1$. Then $E(f) = A_f$ describes the object map of a left adjoint to $\hat{T}$ (and g_f is the corresponding component of the unit of this adjunction).
($\Leftarrow$): Let E be a left adjoint to $\hat{T}$. We must give a strict generic factorisation for any $f : A \to TB$, but it suffices to consider the case $B = 1$, because if $T!g$ is a strict generic factorisation of $T!f$, then $(T\delta)g$ is a strict generic factorisation of f, where δ is the unique T-fill indicated in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & TB \arrow[d, "T!"] \\ TC \arrow[r, "T!"'] \arrow[ur, dotted, "T\delta"] & T1 \end{tikzcd}$$

Observe that for $f : A \to T1$, the component of the unit at f of the given adjunction is

$$\begin{tikzcd} A \arrow[rr, "\eta_f"] \arrow[dr, "f"'] & & TEf \arrow[dl, "T!"] \\ & T1 & \end{tikzcd}$$

and the strict genericness of η_f amounts to the universal property of η as the unit of an adjunction. ∎

5.10. Proposition.

1. *ϕ is weakly cartesian relative to $A \Rightarrow \phi$ preserves generics relative to A.*
2. *ϕ is cartesian relative to $A \Rightarrow \phi$ preserves strict generics relative to A.*
3. *S admits generic factorisations relative to A and ϕ preserves generics relative to A $\Rightarrow \phi$ is weakly cartesian relative to A.*
4. *S admits strict generic factorisations relative to A and ϕ preserves strict generics relative to $A \Rightarrow \phi$ is cartesian relative to A.*
5. *S and T admit strict generic factorisations relative to A and ϕ preserves generics relative to $A \Rightarrow \phi$ is cartesian relative to A.*

Proof. (1): Given a S-generic f, and α, β, and γ so that the outside of

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, dotted, "\delta"'] \arrow[dd, "f"'] & \arrow[d, phantom, "\mathrm{I}"] & T(X) \arrow[dddd, "T(\gamma)"] \\
& S(X) \arrow[ur, "\phi"'] \arrow[dd, "S(\gamma)"'] & \\
S(B) \arrow[ur, phantom, "\mathrm{II}"] \arrow[dr, "S(\beta)"] \arrow[dd, "\phi"'] & & \arrow[l, phantom, "\mathrm{III}"] \\
& S(Y) \arrow[dr, "\phi"] & \\
T(B) \arrow[rr, "T(\beta)"'] & & T(Y)
\end{tikzcd}$$

commutes, the arrow δ is induced so that (I) and (II) commute, since (III) is weakly cartesian relative to A. Since f is S-generic, there is a S-fill ε for (II). By the naturality of ϕ, ε must also be a T-fill for the outer square as required.

(2): Assuming now that f is a strict S-generic and that ϕ is cartesian, it suffices to show that ε obtained above is a unique T-fill. Let ε' be another T-fill. Then since (III) is a pullback, ε' is also a S-fill, and so by the strictness of f, $\varepsilon' = \varepsilon$.

(3): Given α, β and γ making

$$\begin{tikzcd}
A \arrow[drr, bend left, "\beta"] \arrow[dr, dotted] \arrow[ddr, bend right, "\alpha"'] & & \\
& S(X) \arrow[r, "\phi"] \arrow[d, "S(\gamma)"'] & T(X) \arrow[d, "T(\gamma)"] \\
& S(Y) \arrow[r, "\phi"'] & T(Y)
\end{tikzcd}$$

commute, we must provide a commutative fill (dotted arrow). Since S admits generic factorisations relative to A, we can factor α as $S(h) \circ g$ where g is S-generic, to obtain

$$\begin{tikzcd}
A \arrow[d, "g"'] \arrow[rrd, bend left, "\beta"] & & \\
S(H) \arrow[r, dotted, "S(\delta)"] \arrow[rd, "S(h)"'] \arrow[dd, "\phi"'] & S(X) \arrow[r, "\phi"] \arrow[d, "S(\gamma)"] & T(X) \arrow[d, "T(\gamma)"] \\
& S(Y) \arrow[r, "\phi"'] & T(Y) \\
T(H) \arrow[rru, bend right, "T(h)"'] & &
\end{tikzcd}$$

Since ϕ preserves generics relative to A, the composite $\phi_H \circ g$ is T-generic, and so this diagram has a T-fill δ. The desired commutative fill is $S(\delta) \circ g$.
(4): Arguing as in (3) and assuming that g is a strict generic, we must see that the commutative fill $A \longrightarrow S(X)$ is unique. Let ε be such a commutative fill, induce the δ' in

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, "g"'] & S(X) \arrow[d, "S(\gamma)"] \\
S(H) \arrow[r, "S(h)"'] \arrow[ru, dotted, "S(\delta')"] & S(Y)
\end{tikzcd}$$

and so it suffices to show $\delta = \delta'$. This follows since δ' is also a T-fill for the outside of the second diagram of (3), and since ϕ preserves strict generics.
(5): If T admits strict generic factorisations relative to A then all generics $A \rightarrow TB$ are strict, so ϕ in fact preserves strict generics relative to A, and the result follows from (4). ∎

5.11. Proposition. *ϕ is cartesian $\Rightarrow$ ϕ reflects generics and strict generics.*

Proof. Consider f as above so that $\phi_B \circ f$ is (strict) T-generic, and α, β and γ so that the top-left square of

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"'] & S(X) \arrow[r, "\phi"] \arrow[d, "S(\gamma)"] & T(X) \arrow[dd, "T(\gamma)"] \\
S(B) \arrow[r, "S(\beta)"] \arrow[d, "\phi"'] & S(Y) \arrow[rd, "\phi"] & \\
T(B) \arrow[rr, "T(\beta)"'] & & T(Y)
\end{tikzcd}$$

commutes. Then the cartesianness of ϕ guarantees that a (unique) T-fill for the whole square is a (unique) S-fill for the top-left square. ∎

5.12. Lemma. *ϕ preserves generics and generics for T are strict $\Rightarrow$ generics for S are strict.*

Proof. Consider the diagram of the previous proposition this time with ϕ generic-preserving and f generic. An S-fill for the top-left square is a T-fill for the large square, which is unique since $\phi \circ f$ is strict generic. ∎

5.13. Proposition. *Let ϕ be weakly cartesian relative to A.*

1. *S admits generic factorisations relative to $A \Rightarrow \phi$ reflects generics relative to A.*
2. *T admits generic factorisations relative to A and ϕ reflects generics relative to A $\Rightarrow S$ admits generic factorisations relative to A.*
3. *T admits strict generic factorisations relative to A and ϕ reflects generics relative to $A \Rightarrow S$ admits strict generic factorisations relative to A and ϕ is cartesian relative to A.*

Proof. (1): Suppose that the composite

$$A \xrightarrow{f} S(X) \xrightarrow{\phi_X} T(X)$$

is T generic. Factor f as $S(h) \circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & S(X) \arrow[r, "\phi_X"] & T(X) \\ & S(H) \arrow[u, "S(h)"'] \arrow[r, "\phi_H"'] & T(H) \arrow[u, "T(h)"'] \end{tikzcd}$$

where g is S-generic, so that $\phi_H \circ g$ is T-generic by lemma(5.10). Thus, by lemma(5.7), h is an isomorphism, so that f is S-generic.
(2): Given $f : A \rightarrow SZ$, factorise $\phi_Z \circ f$ to obtain

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g'"] \arrow[ddr, bend right, "g"'] & S(Z) \arrow[r, "\phi_Z"] & T(Z) \\ & S(H) \arrow[u, "S(h)"'] \arrow[d, "\phi_H"] & \\ & T(H) \arrow[uur, bend right, "T(h)"'] & \end{tikzcd}$$

where g is T-generic. We induce g' from the weak cartesianness of ϕ relative to A, and g' is S-generic since ϕ reflects generics relative to A.
(3): Immediate from (2), lemma (5.12) and proposition (5.10)(5). ∎

5.14. LEMMA. *Let $f : A \to SB$ be (strict) S-generic and $g : B \to TC$ be (strict) T-generic. Then the composite*

$$A \xrightarrow{f} S(B) \xrightarrow{S(g)} ST(C)$$

is (strict) ST-generic.

PROOF. Suppose first that f and g are generics. An ST-fill for

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & ST(X) \arrow[dd, "T(\gamma)"] \\ S(B) \arrow[d, "S(g)"'] & \\ ST(C) \arrow[r, "ST(\beta)"'] & ST(Y) \end{tikzcd}$$

is obtained as a T-fill for the left square

$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[d, "g"'] & T(X) \arrow[d, "T(\gamma)"] \\ T(C) \arrow[r, "T(\beta)"'] & T(Y) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & ST(X) \arrow[d, "S(T(\gamma))"] \\ S(B) \arrow[r, "S(T(\beta)\circ g)"'] & ST(Y) \end{tikzcd}$$

where δ is an S-fill for right square. Now suppose that f and g are strict generics. Given an ST-fill ε, $S(\varepsilon) \circ g$ is an S-fill for the right square, thus $S(\varepsilon) \circ g = \delta$ since f is strict generic, and so ε is the unique T-fill for the left square since g is strict generic, whence ε is the unique ST-fill. ∎

5.15. COROLLARY. *If S admits generic factorisations relative to A and T admit generic factorisations, then ST admits generic factorisations relative to A.*

5.16. PROPOSITION.

1. *T admits generic factorisations relative to $A \Rightarrow T$ takes wide weak pullbacks to wide squares that are weakly cartesian relative to A.*

2. *T admits strict generic factorisations relative to $A \Rightarrow T$ takes wide pullbacks to wide squares that are cartesian relative to A.*

PROOF. (1): Let

$$((\pi_i : W \to B_i, f_i : B_i \to C) : i \in I)$$

be a weak wide pullback, and

$$((a_i : A \to TB_i, Tf_i : TB_i \to TC) : i \in I)$$

be a wide commutative square. Choose $i \in I$ and factor a_i as

$$A \xrightarrow{g} T(Y) \xrightarrow{T(h_i)} T(B_i)$$

where g is T-generic. Then for $j \in I$ where $j \neq i$, induce h_j as a T-fill of

$$\begin{tikzcd} A \arrow[r, "a_j"] \arrow[d, "g"'] & T(B_j) \arrow[d, "T(f_j)"] \\ T(Y) \arrow[r, "T(f_i h_i)"'] & T(C) \end{tikzcd}$$

and so we have a wide commutative square

$$((h_j : Y \to B_j, f_j : B_j \to C) : j \in I)$$

and since the original wide square is a weak wide pullback, we have $y : Y \to W$ so that $\pi_j y = h_j$ for $j \in I$. Thus $T(y)g$ provides the desired fill exhibiting

$$((\,T(W) \xrightarrow{T(\pi_j)} T(B_j)\,,\ T(B_j) \xrightarrow{T(f_j)} T(C)\,) : j \in I)$$

as weakly cartesian relative to A.
(2): Suppose that the original wide square described above is cartesian. Let $b : A \to TW$ be a filler, that is, $T(\pi_j)b = a_j$ for $j \in I$. We must show that $b = T(y)g$. Let δ_j be the T-fill of

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "g"'] & T(W) \arrow[d, "T(f_j)"] \\ T(Y) \arrow[r, "T(h_j)"'] & T(B_j) \end{tikzcd}$$

but composing this square with f_j, δ_j is also the T-fill of

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "g"'] & T(W) \arrow[d, "T(f)"] \\ T(Y) \arrow[r, "T(h)"'] & T(C) \end{tikzcd}$$

and so is independent of j (that is $\delta_j = \delta_k$ for $j, k \in I$) since T-generics are strict. We denote this common fill by δ. Since δ satisfies $\pi_j \delta = h_j$ for $j \in I$, $\delta = y$ by the uniqueness of y, and so $b = T(y)g$ as required. ∎

An immediate consequence of this last result is

5.17. Corollary.

1. *T admits generic factorisations $\Rightarrow$ T preserves weak wide pullbacks.*

2. *T admits strict generic factorisations $\Rightarrow$ T preserves wide pullbacks.*

6. Further exactness results

Our purpose in this section is to understand two results which are true given additional hypotheses on $\mathcal{A}$ and T. These results are

1. Theorem(6.6) – which is a converse to (5.17)(2).

2. Theorem(6.8) – by which the admission of (not necessarily strict) generic factorisations implies the preservation of certain special connected limits. Namely, the preservation of *cofiltered* limits.

The additional hypotheses alluded to above will be drawn from

6.1. Definition. *Let $I : \mathcal{S} \to \mathcal{A}$ be the inclusion of a small full subcategory $\mathcal{S}$. For $A \in \mathcal{A}$, denote by*

$$\begin{tikzcd} \mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & 1 \arrow[d, "A"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd}$$

the usual comma square. That is, an object of $\mathcal{S}{\downarrow}A$ is an arrow $f : S \to A$ where $S \in \mathcal{S}$, and $\alpha_f = f$. We shall refer to α as "A's canonical diagram". $\mathcal{A}$ is accessible from $\mathcal{S}$ *when*

1. *For $A \in \mathcal{A}$ and $F : \mathcal{S}{\downarrow}A \to \mathcal{A}$, the colimit of F exists.*

2. *For $A \in \mathcal{A}$, A's canonical diagram is a left extension (that is, a universal cocone).*

An object $B \in \mathcal{A}$ is $\mathcal{S}$-small *when for $f : B \to C$ and*

$$\begin{tikzcd} \mathcal{S}{\downarrow}\mathcal{A} \arrow[rr, "u"] \arrow[dr, "F"'] \arrow[rr, phantom, shift right=4ex, "\overset{\psi}{\Rightarrow}"] & & 1 \arrow[dl, "C"] \\ & \mathcal{A} & \end{tikzcd}$$

exhibiting C as a left extension of F along u, there is $g : S \to A$ so that f factors as

$$B \longrightarrow F(g) \longrightarrow C$$

We say that $\mathcal{S}$ is retract-closed *when a retract of any object in $\mathcal{S}$ is isomorphic to an object in S. T is said to be* $\mathcal{S}$-ranked *when it preserves colimits of all $F : \mathcal{S}{\downarrow}A \to \mathcal{A}$.*

6.2. Remarks.

1. Condition(2) says that the inclusion is a dense functor in the usual sense, and so the condition that $\mathcal{A}$ is accessible from $\mathcal{S}$ amounts to $\mathcal{S}$ being a dense subcategory together with a cocompleteness condition on $\mathcal{A}$. It is instructive to unpack the density condition. It says that to specify a morphism $A \to B$ in $\mathcal{A}$, one must assign to each $a : S \to A$ an arrow $fa : S \to B$ in such a way that $a'g = a \Rightarrow (fa')g = fa$. That is, maps out of $S \in \mathcal{S}$ play the role of generalised elements.

2. A sufficient condition for B to be $\mathcal{S}$-small is for $\mathcal{A}(B, -)$ to preserve the colimits of all $F : \mathcal{S}{\downarrow}A \to \mathcal{A}$.

6.3. Examples.

1. Fix a regular cardinal λ and denote by $\mathcal{A}_\lambda$ the full subcategory of $\mathcal{A}$ consisting of the λ-presentable objects. If $\mathcal{A}$ is λ-accessible in the usual sense then $\mathcal{A}_\lambda$ is essentially small (that is, equivalent to a small category) and $\mathcal{A}$ is accessible from $\mathcal{A}_\lambda$. λ-presentable objects are $\mathcal{A}_\lambda$-small by 6.2(2). If T preserves λ-filtered colimits then it is $\mathcal{A}_\lambda$-ranked.

2. Let $\mathbf{C}$ be a small category. $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is $\mathbf{C}$-accessible via the yoneda embedding. Representable functors are $\mathbf{C}$-small by 6.2(2) since homming out of $\mathbf{C}(-, C)$ is evaluating at C, which has left and right adjoints, and so in particular preserves all colimits.

6.4. Definition. *Given $f : A \to T(B)$, the category* $\mathrm{Fact}(f)$ *is defined as follows:*

- *objects: pairs $(g : A \to T(C), h : C \to B)$ such that $f = T(h)g$.*
- *arrows: an arrow $(g, h) \to (g', h')$ in* $\mathrm{Fact}(f)$ *is a morphism δ as in*

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "g"'] & T(C') \arrow[d, "T(h')"] \\ T(C) \arrow[r, "T(h)"'] \arrow[ur, dotted, "T(\delta)"] & T(Y) \end{tikzcd}$$

such that $\delta T(g) = T(g')$ and $h'\delta = h$.

Clearly to demand that T admits generic factorisations is to ask that $\mathrm{Fact}(f)$ has a weak initial object for every such f. Similarly, T admits strict generic factorisations if and only if $\mathrm{Fact}(f)$ has an initial object for every such f.

6.5. Lemma. *If $\mathcal{A}$ has wide pullbacks and T preserves them, then for any $f : A \to T(B)$,* $\mathrm{Fact}(f)$ *is complete.*

Proof. It suffices to show that $\mathrm{Fact}(f)$ has (arbitrary) products and pullbacks, and these are easily constructed directly using the hypotheses of this lemma. ∎

Recall from [Mac71], that a *solution set* for a category $\mathcal{A}$ is a set $\mathcal{X}$ of objects of $\mathcal{A}$ that is jointly weakly initial, in the sense that for any $A \in \mathcal{A}$, there is an $S \in \mathcal{X}$ and a morphism $S \to A$. Recall also that the initial object theorem constructs an initial object for any category that is locally small, complete, and possesses a solution set. Clearly such an initial object will be a retract of some $S \in \mathcal{X}$.

6.6. THEOREM. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$, $\mathcal{S}$ be retract closed, all $S \in \mathcal{S}$ be $\mathcal{S}$-small, and T be $\mathcal{S}$-ranked. If T preserves wide pullbacks, then it admits strict generic factorisations.*

PROOF. For any $f : A \to TB$ we must construct an initial object for $\mathrm{Fact}(f)$. By the initial object theorem it suffices by (6.5) to verify that $\mathrm{Fact}(f)$ has a solution set. We shall show that there is a set $\mathcal{X}$ of objects of $\mathcal{A}$, depending only on A, such that any f as above factors as

$$A \xrightarrow{g} T(S) \xrightarrow{T(h)} T(B)$$

where $S \in \mathcal{X}$ A solution set for $\mathrm{Fact}(f)$ will be obtained as consisting of pairs

$$(\ A \longrightarrow T(S)\ ,\ S \longrightarrow B\ )$$

where $S \in \mathcal{X}$. Since $\mathcal{X}$ is a set and $\mathcal{A}$ is locally small, this is indeed a set. Consider first the case where A is in $\mathcal{S}$. Noting that $T(B)$ is the colimit of T applied to B's canonical diagram, since T is $\mathcal{S}$-ranked, the $\mathcal{S}$-smallness of A guarantees that f factors through a component of this colimiting cocone for $T(B)$. Thus we can take the elements of $\mathcal{X}$ to be the objects of $\mathcal{S}$. An initial object of $\mathrm{Fact}(f)$, which is a generic factorisation of f, is a retract of a factorisation of f in our solution set, which amounts to saying that f factors as

$$A \xrightarrow{g} T(S) \xrightarrow{T(h)} T(B)$$

where g is generic, and S is the retract of an object $\mathcal{S}$, and thus can be taken to be in $\mathcal{S}$ since $\mathcal{S}$ is retract closed. In the general case, for any $S \in \mathcal{S}$ and $\sigma \in \mathcal{A}(S, A)$, using the $\mathcal{S}$-smallness of S and the rank of T, we obtain $S_\sigma \in \mathcal{S}$, g_σ and h_σ making

$$\begin{tikzcd}
S \arrow[r, "\sigma"] \arrow[d, "g_\sigma"'] & A \arrow[d, "f"] \\
T(S_\sigma) \arrow[r, "T(h_\sigma)"'] & T(B)
\end{tikzcd}$$

commute. Moreover, we may assume that the g_σ are generic by the above analysis of the case where $A \in \mathcal{S}$ (applied to $A = S$). Given an arrow

$$\begin{tikzcd}
S_1 \arrow[rr, "\psi"] \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\
& A &
\end{tikzcd}$$

in $\mathcal{S}{\downarrow}A$, notice that there is a unique T-fill S_ψ in

$$\begin{tikzcd}
S_1 \arrow[r, "\psi"] \arrow[d, "g_{\sigma_1}"'] & S_2 \arrow[r, "g_{\sigma_2}"] & T(S_{\sigma_2}) \arrow[d, "T(h_{\sigma_2})"] \\
T(S_{\sigma_1}) \arrow[urr, dotted, "T(S_\psi)"] \arrow[rr, "T(h_{\sigma_1})"'] & & T(B)
\end{tikzcd}$$

as g_{σ_1} is generic. Since these T-fills are induced uniquely, they provide the arrow mappings of a functor $S_{(-)}$, and the g_σ's and the h_σ's are the components of natural transformations g and h, so that

$$\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[rr] \arrow[dr, "S_{(-)}"] \arrow[dd, "\pi"'] & & 1 \arrow[dr, "B"] \arrow[d, phantom, "\Rightarrow h"] & \\
& \mathcal{S} \arrow[rr, "I"] \arrow[r, phantom, "\Rightarrow g"] & & \mathcal{A} \arrow[dl, "T"] \\
\mathcal{S} \arrow[rr, "I"'] & & \mathcal{A} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Rightarrow k_A"] & 1 \arrow[d, "A"'] \arrow[dr, "B"] & \\
\mathcal{S} \arrow[r, "I"'] & \mathcal{A} \arrow[r, phantom, "\Rightarrow f"] & \mathcal{A} \arrow[l, "T"]
\end{tikzcd}$$

Upon taking colimits, and denoting by $\overline{S}$ the colimit of $IS_{(-)}$, this last equality provides a factorisation of f as

$$A \xrightarrow{\overline{g}} T(\overline{S}) \xrightarrow{T(\overline{h})} T(B)$$

So we can take $\mathcal{X}$ to be objects in the image of

$$[\mathcal{S}{\downarrow}A, \mathcal{S}] \xrightarrow{I\circ -} [\mathcal{S}{\downarrow}A, \mathcal{A}] \xrightarrow{\text{colim}} \mathcal{A}$$

which is small because the functor category $[\mathcal{S}{\downarrow}A, \mathcal{S}]$ is small, since $\mathcal{S}{\downarrow}A$ and $\mathcal{S}$ are small. ■

In the previous proof it was necessary to understand generic factorisations of morphisms out of objects of $\mathcal{S}$ first, before proceeding to the general case. This made evident the following useful fact.

6.7. Scholium. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$, $\mathcal{S}$ be retract-closed, all $S \in \mathcal{S}$ be $\mathcal{S}$-small and T be $\mathcal{S}$-ranked. If $A \in \mathcal{S}$ and $g : A \to TB$ is generic, then B is isomorphic to an object of $\mathcal{S}$.*

Proof. T applied to B's canonical diagram gives TB as a colimit of TS's where $S \in \mathcal{S}$, and A is $\mathcal{S}$-small, and so g factors through a component of this universal cocone. This exhibits B as a retract of some $S \in \mathcal{S}$. ■

Recall that a category is *cofiltered* when its dual is filtered, and a functor F is *cofinitary* when it preserves cofiltered limits, or equivalently, when F^{op} is finitary.

6.8. Theorem. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$, $\mathcal{S}$ be retract-closed, all $S \in \mathcal{S}$ be $\mathcal{S}$-small and T be $\mathcal{S}$-ranked. Suppose also that the automorphism groups of the $S \in \mathcal{S}$ are finite. If T admits generic factorisations relative to the objects of $\mathcal{S}$ then T is cofinitary.*

Proof. By the dual of theorem(1.5) and corollary(1.7) of [AR94], it suffices to show that T preserves limits of diagrams of the form

$$\cdots \xrightarrow{\partial_2} X_2 \xrightarrow{\partial_1} X_1 \xrightarrow{\partial_0} X_0$$

Let $c_n : X \to X_n$ be the components of a universal cone. We must show that the cone for the diagram

$$\cdots \xrightarrow{T(\partial_2)} T(X_2) \xrightarrow{T(\partial_1)} T(X_1) \xrightarrow{T(\partial_0)} T(X_0)$$

whose components are $T(c_n)$, is universal. Let $x_n : A \to TX_n$ be the components of some other cone for this diagram. We must show that there is a unique $x : A \to TX$ such that $T(c_n) \circ x = x_n$. By 6.2(1) it suffices to consider the case where $A \in \mathcal{S}$. Factor x_0 as

$$A \xrightarrow{g} T(B) \xrightarrow{T(h_0)} T(X_0)$$

where g is T-generic and $B \in \mathcal{S}$ by (6.7). By induction on $n \in \mathbb{N}$, assume that we have factored x_n as

$$A \xrightarrow{g} T(B) \xrightarrow{T(h_n)} T(X_n)$$

so that we have

$$\begin{tikzcd} A \arrow[r, "x_{n+1}"] \arrow[d, "g"'] & T(X_{n+1}) \arrow[d, "T(\partial_n)"] \\ T(B) \arrow[r, "T(h_n)"'] & T(X_n) \end{tikzcd}$$

commutative. A T-fill h_{n+1} for this diagram provides the analogous factorisation of x_{n+1}, and satisfies $\partial_n \circ h_{n+1} = h_n$. Thus the h_n provide the components of a cone for the original diagram, and so there is a unique h for which $c_n \circ h = h_n$. We can take x to be $T(h) \circ g$. To see that this x is unique, consider x' so that $T(c_n) \circ x' = x_n$. Factor x' as

$$A \xrightarrow{g'} T(B') \xrightarrow{T(h')} T(X)$$

where g' is T-generic and $B' \in \mathcal{S}$, and observe that for each $n \in \mathbb{N}$,

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "g"'] & T(B') \arrow[d, "T(c_n \circ h')"] \\ T(B) \arrow[r, "T(h_n)"'] & T(X_n) \end{tikzcd}$$

is commutative. Since g is T-generic, each of these squares must have a T-fill. Observe that if $\delta \in \mathcal{A}(B, B')$ is a T-fill for the n-th square, then it is a T-fill for the m-th square when $m < n$. Thus either δ is a T-fill for every such square, or there is a maximum n for which it T-fills the n-th square. Since B and $B' \in \mathcal{S}$, the set of isomorphisms in $\mathcal{A}(B, B')$ is finite. Since there are infinitely many squares each of which are T-filled by some $\delta \in \mathcal{A}(B, B')$, there must exist a δ which T-fills every square. For this particular δ, $T(\delta) \circ g = g'$ and for $n \in \mathbb{N}$, $c_n \circ h' \circ \delta = h_n = c_n \circ h$. Since the c_n form a universal cone, they are jointly monic, so that $h' \circ \delta = h$, and so $x' = T(h') \circ g' = T(h) \circ g = x$. ∎

7. Parametric representability

We shall now explain the sense in which endofunctors T that admit strict generic factorisations have a family of representing objects. In doing so, we obtain 2-categorical descriptions of those concepts related to strict generic morphisms.

7.1. Definition. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$. A* parametric representation for T from $\mathcal{S}$ *is a 2-cell*

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, "E_T"] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\overset{g_T}{\Rightarrow}"] & \mathcal{A} \arrow[d, "T"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd}$$

that satisfies the following universal property: given any 2-cell ϕ as shown below, there is $\tilde{\phi}$ and $\overline{\phi}$ unique so that $F = \pi_T \tilde{\phi}$ and

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[dd, "F"'] & \mathcal{B} \arrow[d, "H"] \\ & \mathcal{A} \arrow[d, "T"] \arrow[l, phantom, "\overset{\phi}{\Rightarrow}"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "\tilde{\phi}"'] \arrow[dr, phantom, "\overset{\overline{\phi}}{\Rightarrow}"] & \mathcal{B} \arrow[d, "H"] \\ \mathcal{E}_T \arrow[r, "E_T"] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\overset{g_T}{\Rightarrow}"] & \mathcal{A} \arrow[d, "T"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd} \qquad (2)$$

The functor E_T is called the exponent *of the parametric representation. A functor T is said to be* parametrically representable from $\mathcal{S}$ *when there is such a parametric representation for it. It is straightforward to verify that the above universal property determines π_T, $\mathcal{E}_T$ and E_T up to isomorphism.*

7.2. Example. For $n \in \mathbb{N}$, let $g_n : 1 \to \mathcal{M}n$ pick out $1_n \in \mathcal{M}n$. Then this defines a parametric representation

$$\begin{tikzcd} \mathbb{N} \arrow[r, "E_{\mathcal{M}}"] \arrow[d] \arrow[dr, phantom, "\overset{g}{\Rightarrow}"] & \mathbf{Set} \arrow[d, "\mathcal{M}"] \\ 1 \arrow[r, "1"'] & \mathbf{Set} \end{tikzcd}$$

of $\mathcal{M}$ from 1.

Given a parametric representation for T another description of $f : A \to TB$ becomes available. By the definition of parametric representability, there is $\tilde{f}$ and $\overline{f}$ unique so that $\pi_T \tilde{f} = \pi_A$ and

$$
\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & 1 \arrow[d, "A"'] \arrow[d, bend left=60, "TB"] \arrow[d, phantom, shift left=3, "\overset{f}{\Rightarrow}"] \\
\mathcal{S} \arrow[r, "I"'] & \mathcal{A}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & 1 \arrow[d, "B"] \\
\mathcal{E}_T \arrow[r, "E_T"] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\overset{g_T}{\Rightarrow}"] & \mathcal{A} \arrow[d, "T"] \\
\mathcal{S} \arrow[r, "I"'] & \mathcal{A}
\end{tikzcd}
\tag{3}
$$

and since α is a left extension, $\tilde{f}$ and $\overline{f}$ determine f uniquely as well.

7.3. Proposition. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$ and T be parametrically representable from $\mathcal{S}$. Then $\overline{f}$ is a left extension iff f is strict T-generic. Moreover, T admits strict generic factorisations.*

Proof. To say that

$$
\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "f"'] & TX \arrow[d, "T\gamma"] \\
TB \arrow[r, "T\beta"'] & TY
\end{tikzcd}
$$

commutes is to say that

$$
\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & 1 \arrow[d, "B"'] \arrow[d, bend left=60, "Y"] \arrow[d, phantom, shift left=3, "\overset{\beta}{\Rightarrow}"] \\
\mathcal{E}_T \arrow[r, "E_T"'] & \mathcal{A}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\overset{\overline{\phi}}{\Rightarrow}"] & 1 \arrow[d, "X"'] \arrow[d, bend left=60, "Y"] \arrow[d, phantom, shift left=3, "\overset{\gamma}{\Rightarrow}"] \\
\mathcal{E}_T \arrow[r, "E_T"'] & \mathcal{A}
\end{tikzcd}
$$

so a T-fill for the original square is induced uniquely when $\overline{f}$ is a left extension. Since $\mathcal{A}$ is accessible from $\mathcal{S}$, any such $\overline{f}$ will factor as

$$
\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\overset{\overline{g}}{\Rightarrow}"] & 1 \arrow[d, "C"'] \arrow[d, bend left=60, "B"] \arrow[d, phantom, shift left=3, "\overset{h}{\Rightarrow}"] \\
\mathcal{E}_T \arrow[r, "E_T"'] & \mathcal{A}
\end{tikzcd}
$$

where $\overline{g}$ is a left extension, and so $f = T(h)g$ with $\overline{g}$ a left extension. Thus T admits strict generic factorisations. Taking f to be generic, h is an isomorphism by (5.7), whence $\overline{f}$ is a left extension. ∎

7.4. EXAMPLES.

1. Take $\mathcal{A}$ to be **Set** and $\mathcal{S}$ to consist of the one-element set 1. If T is parametrically representable with $\mathcal{E}_T = 1$ then T is representable in the usual sense, with the exponent E_T picking out the representing object of T.

2. For $\mathcal{S}$ and $\mathcal{A}$ as in (1), T is parametrically representable iff T is a coproduct of representable functors, and $\mathcal{E}_T$ must always be a set, with E_T picking out the representing objects of these representable functors.

7.5. THEOREM. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$ and have a terminal object* 1. *If T admits strict generic factorisations relative to the objects of $\mathcal{S}$ then it is parametrically representable from $\mathcal{S}$. Moreover, the resulting parametric representation exhibits T as a pointwise left extension of $I\pi_T$ along E_T, and π_T is a discrete fibration.*

PROOF. For $S \in \mathcal{S}$ and $x : S \to T1$, choose a strict generic factorisation

$$S \xrightarrow{g_x} T\overline{x} \xrightarrow{T(!)} T1$$

of x. Given

$$\begin{tikzcd} S_1 \arrow[d, "f"'] \arrow[dr, "x"] & \\ S_2 \arrow[r, "y"'] & T1 \end{tikzcd}$$

an arrow of $\mathcal{S}\downarrow T1$, the strict genericness of g_x induces a unique $\overline{f_{xy}} : \overline{x} \to \overline{y}$ making

$$\begin{tikzcd} S_1 \arrow[r, "f"] \arrow[d, "g_x"'] & S_2 \arrow[d, "g_y"] \\ T\overline{x} \arrow[r, "T\overline{f_{xy}}"'] & T\overline{y} \end{tikzcd}$$

commute. So we define that $E_T : \mathcal{S}\downarrow T1 \to \mathcal{A}$ takes the above arrow of $\mathcal{S}\downarrow T1$ to $\overline{f_{xy}}$, and the component of

$$\begin{tikzcd} \mathcal{S}\downarrow T1 \arrow[r, "E_T"] \arrow[d, "\pi_{T1}"'] \arrow[dr, phantom, "\overset{g}{\Rightarrow}"] & \mathcal{A} \arrow[d, "T"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd}$$

at x as g_x. It now remains to verify that g is a parametric representation for T and that it exhibits T as a pointwise left extension of $I\pi_{T1}$ along E_T. To see that G is a parametric representation, consider ϕ as in (2). For $C \in \mathcal{C}$ take $\tilde{\phi}(C) = T!\phi_C$ and $\overline{\phi}_C$ to be the unique T-fill of

$$\begin{tikzcd} IFC \arrow[r, "\phi_C"] \arrow[d, "g_x"'] & THGC \arrow[d, "T!"] \\ T\overline{x} \arrow[r, "T!"'] & T1 \end{tikzcd}$$

where $x = \tilde{\phi}(C)$. For $f : C_1 \to C_2$ in $\mathcal{C}$, take $\tilde{\phi}(f) = IFf$. Notice that $F = \pi_{T1}\tilde{\phi}$ and (2) hold by definition. The functoriality of $\tilde{\phi}$, the naturality of $\overline{\phi}$ and their required uniqueness follows easily since all the g_x's are *strict* generics.
To see that g is a pointwise left extension we verify directly that given ψ and X

$$\begin{tikzcd} \mathcal{E}_T{\downarrow}X \arrow[r] \arrow[d] & \mathcal{B} \arrow[dd, "U"] \\ \mathcal{S}{\downarrow}T1 \arrow[r, phantom, "\overset{\psi}{\Longrightarrow}"] \arrow[d, "\pi_{T1}"'] & \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{E}_T{\downarrow}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "X"] \arrow[dd, bend left=60, "U"] \\ \mathcal{S}{\downarrow}T1 \arrow[r, "E_T"] \arrow[d, "\pi_{T1}"'] \arrow[dr, phantom, "\overset{g}{\Longrightarrow}"] & \mathcal{A} \arrow[d, "T"] \arrow[r, phantom, "\overset{\hat{\psi}}{\Rightarrow}"] & {} \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd} \tag{4}$$

there is a unique $\hat{\psi}$ satisfying (4). Here λ is the usual comma object, that is, the natural transformation whose component at $(x : S \to T1, y : E_T(x) \to XB, B)$ is y. For $B \in \mathcal{B}$, in view of (6.2)(1), we specify $\hat{\psi}_B : TXB \to UB$ by assigning to each $f : S \to TXB$ with $S \in \mathcal{S}$, the morphism $\psi_{(}x, \hat{f}, B) : S \to UB$, where $x = T!(f)$ and $\hat{f}$ is the unique T-fill for

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g_x"'] & TXB \arrow[d, "T!"] \\ T\overline{x} \arrow[r, "T!"'] & T1 \end{tikzcd}$$

The well-definedness of the $\hat{\psi}_B$, their naturality, equation(4), and the required uniqueness of $\hat{\psi}$ are all straightforward to verify. ∎

When $\mathcal{A}$ is accessible from $\mathcal{S}$, and T and U are parametrically representable from $\mathcal{S}$, another description of 2-cells $\phi : T \Rightarrow U$ becomes available. That is, there is $\tilde{\phi}$ and $\overline{\phi}$ unique so that $\pi_U\tilde{\phi} = \pi_T$ and

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, "E_T"] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\overset{g_T}{\Longrightarrow}"] & \mathcal{A} \arrow[d, "T"'] \arrow[d, bend left=60, "U"] \arrow[d, phantom, shift left=12, "\overset{\phi}{\Rightarrow}"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{E}_T \arrow[d, "\tilde{\phi}"'] \arrow[dr, "E_T"] \arrow[d, phantom, shift left=10, "\overset{\Rightarrow}{\overline{\phi}}"] & \\ \mathcal{E}_U \arrow[r, "E_U"] \arrow[d, "\pi_U"'] \arrow[dr, phantom, "\overset{g_U}{\Longrightarrow}"] & \mathcal{A} \arrow[d, "U"] \\ \mathcal{S} \arrow[r, "I"'] & \mathcal{A} \end{tikzcd} \tag{5}$$

and when g_T is a left extension, as is the case when $\mathcal{A}$ has a terminal object, $\tilde{\phi}$ and $\overline{\phi}$ determine ϕ uniquely as well.

7.6. Theorem. *Let $\mathcal{A}$ be accessible from $\mathcal{S}$ and $\phi : T \Rightarrow U$ with T and U parametrically representable from $\mathcal{S}$.*

1. *$\overline{\phi}$ is an isomorphism $\Rightarrow$ ϕ is cartesian.*
2. *$\mathcal{A}$ has a terminal object and ϕ is cartesian $\Rightarrow$ $\overline{\phi}$ is an isomorphism.*

Proof. (1): By (5.10)(5) and (7.3) it suffices to prove that ϕ preserves generics. Let $f : A \to TB$ be generic. Then by equations (3) and (5), $\overline{\phi_B f}$ is the composite

$$\begin{tikzcd}
\mathcal{S}{\downarrow}A \arrow[r, "!"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\overline{f}\,\Rightarrow"] & 1 \arrow[dd, "B"] \\
\mathcal{E}_T \arrow[d, "\tilde{\phi}"'] \arrow[dr, "E_T"] \arrow[d, phantom, shift left=3ex, "\overline{\phi}\,\Rightarrow"] & \\
\mathcal{E}_U \arrow[r, "E_U"'] & \mathcal{A}
\end{tikzcd} \tag{6}$$

Since f is generic, $\overline{f}$ exhibits B as a left extension of $E_T\tilde{f}$ along !, and since $\overline{\phi}$ is an isomorphism, (6) exhibits B as a left extension of $E_U\tilde{\phi}\tilde{f}$ along !. Thus by (7.3) $\phi_B f$ is generic.

(2): Since $\mathcal{A}$ has a terminal object, the explicit descriptions of the parametric representations of T and U in terms of generics, used in the proof of (7.5), become available. Take $\tilde{\phi}$ to be $\phi_1 \circ - : \mathcal{S}{\downarrow}T1 \to \mathcal{S}{\downarrow}U1$, and for $x : S \to T1$, take $\overline{\phi}_x$ to be the unique U-fill of

$$\begin{tikzcd}
S \arrow[r, "g_x"] \arrow[d, "g_{\phi_1 x}"'] & T\overline{x} \arrow[r, "\phi_{\overline{x}}"] & U\overline{x} \arrow[d, "U!"] \\
U\overline{\phi_1 x} \arrow[rr, "U!"'] & & U1
\end{tikzcd}$$

It is straightforward to verify that this defines $\tilde{\phi}$ and $\overline{\phi}$ satisfying equation(5). Since ϕ preserves generics, $\overline{\phi}_x$ must be an isomorphism for all x by (5.7). ∎

8. Parametrically representable endofunctors of presheaf categories

Let λ be a regular cardinal. Recall that T is said *to have rank* λ when it preserves λ-filtered colimits. We shall now use the above analysis to understand parametric representability for endofunctors of $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.

8.1. THEOREM. *For an endofunctor T of $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ with rank λ, the following statements are equivalent.*

1. *T is parametrically representable from $\mathbf{C}$ (via the yoneda embedding).*
2. *T admits strict generic factorisations.*
3. *T preserves wide pullbacks.*

Moreover when T satisfies these hypotheses, $f : A \to TB$ is generic and A is a λ-presentable object $\Rightarrow$ B is a λ-presentable object.

PROOF. (1) $\Rightarrow$ (2) by (7.3), (2) $\Rightarrow$ (1) by (7.5), and (2) $\Rightarrow$ (3) by (5.17)(2). For (3) $\Rightarrow$ (2) apply (6.6) this time where $\mathcal{S} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]_\lambda$. The final statement of the theorem follows from (6.7). ∎

Applying (7.6) in this case enables one to understand cartesian transformations between such endofunctors of $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.

9. The tree monad and Ω

Two of the most fundamental mathematical objects that arise in the higher category theory of Michael Batanin in [Bat98], [Bat02b], [Bat02a] and [Bat03] are

1. The monad $\mathcal{T}$ on **Glob** whose algebras are strict ω-categories. It was denoted as D_s in [Bat98], and we shall refer to it as the tree monad.
2. The category Ω of plane trees described in (4).

We shall describe these objects in terms of generic morphisms.

The underlying endofunctor of $\mathcal{T}$ is obtained as a left extension

$$\begin{tikzcd} y{\downarrow}\mathbf{Tr} \arrow[r, "E_{\mathcal{T}}"] \arrow[d, "\pi_{\mathcal{T}}"'] \arrow[dr, phantom, "\overset{g_{\mathcal{T}}}{\Rightarrow}"] & \mathbf{Glob} \arrow[d, "\mathcal{T}"] \\ \mathbb{G} \arrow[r, "y"'] & \mathbf{Glob} \end{tikzcd}$$

which in turn is a parametric representation of $\mathcal{T}$. The following proposition is an immediate consequence of this description of $\mathcal{T}$, (4.7) and (7.3).

9.1. PROPOSITION.

1. *$\mathcal{T}$ admits strict generic factorisations.*
2. *$f : A \to \mathcal{T}B$ is generic and A is a globular cardinal $\Rightarrow$ B is a globular cardinal.*

We shall now describe the monad structure purely in terms of generics. This description hinges on the fact that for a globular cardinal A (whose corresponding tree is) of height $\leq n$, there is a unique $x : \mathbb{G}(-,n) \to \mathbf{Tr}$ so that $E_{\mathcal{T}}(x) \cong A$. By (7.3) it follows that this amounts to saying that $f : A \to X$ uniquely determines $\hat{f}$ so that

$$\begin{tikzcd} \mathbb{G}(-,n) \arrow[rr, "\hat{f}"] \arrow[dr, "g"'] & & \mathcal{T}X \\ & \mathcal{T}A \arrow[ur, "\mathcal{T}f"'] & \end{tikzcd}$$

commutes where g is $\mathcal{T}$-generic. Conversely, by generically factorising $\hat{f}$ one determines f and A, and this describes the isomorphism of sets

$$\mathbf{Glob}(\mathbb{G}(-,n), \mathcal{T}X) \cong \sum_{T \in \mathbf{Tr}_n} \mathbf{Glob}(E_{\mathcal{T}}(n,T), X)$$

which corresponds to the original description of $\mathcal{T}$ in [Bat98]. The description of $\mathcal{T}$'s monad structure proceeds as follows:

1. Fix a parametric representation for $\mathcal{T}$ from **tr**. In terms of generics this amounts to a choice of generic factorisation of every $A \to \mathcal{T}X$ where A is a globular cardinal.
2. In view of (6.2)(1) we specify $\eta_X : X \to \mathcal{T}X$ for X in **Glob** by assigning $\hat{x}$ to $x : \mathbb{G}(-,n) \to X$.
3. In view of (6.2)(1) we specify $\mu_X : \mathcal{T}\mathcal{T}X \to \mathcal{T}X$ for X in **Glob** by assigning $\hat{x'}$ to $x : \mathbb{G}(-,n) \to \mathcal{T}\mathcal{T}X$, where the $\mathcal{T}\mathcal{T}$-generic factorisation

$$\begin{tikzcd} \mathbb{G}(-,n) \arrow[rr, "x"] \arrow[dr, "g"'] & & \mathcal{T}\mathcal{T}X \\ & \mathcal{T}\mathcal{T}A \arrow[ur, "\mathcal{T}\mathcal{T}x'"'] & \end{tikzcd}$$

   of x is specified by (1) and (5.14), and so by (9.1) A is a globular cardinal.

The well-definedness of the components of η and μ, their cartesian naturality and the monad axioms are all easily verified. That $\mathcal{T}$-algebras are strict-ω-categories is immediate from the definition, if one takes strict-ω-category structures to be defined as a choice of unique composites of globular pasting diagrams. The proof that this corresponds with strict-ω-categories defined by successive enrichment can be found in [Lei00]. The following characterisation of Ω, originally due to Clemens Berger [Ber02], follows from (4.8), (7.3) and the above generic description of the tree monad.

9.2. PROPOSITION. *Ω is isomorphic to the dual of the following subcategory of the Kleisli category of $\mathcal{T}$:*

- *objects are globular cardinals.*
- *an arrow $A \to B$ is a $\mathcal{T}$-generic morphism $A \to \mathcal{T}B$*

10. Analytic endofunctors of **Set** II

In this subsection we continue the discussion of section (3), and characterise the images of G and E in terms of generics.

10.1. Theorem. *For a finitary endofunctor T of* **Set** *the following statements are equivalent.*

1. *T is strongly analytic.*
2. *T admits strict generic factorisations.*
3. *T preserves wide pullbacks.*

Moreover when T satisfies these hypotheses, $f : A \to TB$ is generic and A is a finite set $\Rightarrow$ B is finite.

Proof. Suppose that T is strongly analytic. Then by definition, T comes equipped with a cartesian transformation $\tau : T \Rightarrow \mathcal{M}$. Thus (1)⇒(2) since $\mathcal{M}$ admits strict generic factorisations by (5.5) and τ preserves and reflects generics since it is cartesian. Now suppose that T is parametrically representable from 1. Then for a given parametric representation, E_T must factor through $\mathbf{Set}_f$ since T is finitary. Defining $\tilde{\phi} : T1 \to \mathbb{N}$ by $\tilde{\phi}x = |E_T x|$ and choosing a bijection $\overline{\phi}_x : E_{\mathcal{M}}(|E_T x|) \to E_T x$ produces

$$\begin{tikzcd} T1 \arrow[d, "\tilde{\phi}"'] \arrow[dr, "E_T"] \arrow[d, phantom, shift left=3, "\Rightarrow \overline{\phi}"] & \\ \mathbb{N} \arrow[r, "E_{\mathcal{M}}"'] & \mathbf{Set} \end{tikzcd}$$

which corresponds by (7.6) and the parametric representation for $\mathcal{M}$ given in (7.2), to a cartesian transformation $\phi : T \Rightarrow \mathcal{M}$. The result now follows from (8.1). ■

10.2. Theorem. *For $\phi : S \Rightarrow T$, a natural transformation between strongly analytic functors, the following statements are equivalent:*

1. *ϕ is strongly analytic.*
2. *ϕ preserves and reflects generics.*
3. *ϕ is cartesian.*

Proof. To say that ϕ is strongly analytic is to say that there are cartesian transformations σ and τ making

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\ & \mathcal{M} & \end{tikzcd}$$

commute. Thus (1)⇒(2). Conversely, if ϕ is cartesian and T is strongly analytic, we have $\tau : T \Rightarrow \mathcal{M}$ cartesian and $\sigma = \tau\phi$ exhibits S and ϕ as strongly analytic. Since S admits strict generic factorisations (2)⇔(3) by (5.10) and (5.11). ■

This completes the characterisation of the image of G. We begin with that of E by observing

10.3. LEMMA. *Let $\phi : X \to Y \in [\mathbb{P}, \mathbf{Set}]$. Then $E\phi$ is weakly cartesian.*

PROOF. Let $f : A \to B \in \mathbf{Set}$. We must show that

$$\begin{tikzcd} E(X)(A) \arrow[r, "E(X)(f)"] \arrow[d, "E(\phi)_A"'] & E(X)(B) \arrow[d, "E(\phi)_B"] \\ E(Y)(A) \arrow[r, "E(Y)(f)"'] & E(Y)(B) \end{tikzcd}$$

is weakly cartesian. Consider $[y \in Y(m), g \in A^m] \in E(Y)(A)$ and $[x \in X(n), h \in B^n] \in E(X)(B)$ so that $[y, fg] = [\phi(x), h]$. This says that $m = n$, and that there is $\rho \in \mathrm{Sym}_n$ so that $Y(\rho)(y) = \phi(x)$ and $fg\rho = h$. Since $[y, g] = [Y(\rho)(y), g\rho]$ by definition, we can assume without loss of generality that y and g were chosen so that $y = \phi(x)$ and $fg = h$. Thus $[x, g] \in E(X)(A)$ satisfies $E(\phi)_A[x, g] = [y, g]$ and $E(X)(f)[x, g] = [x, h]$. ∎

10.4. COROLLARY. *For T analytic, c_T as defined in (3) is weakly cartesian.*

We denote by $\mathrm{WcEnd}(\mathbf{Set})$ the subcategory of $\mathrm{End}(\mathbf{Set})$ consisting of all endofunctors of $\mathbf{Set}$ and weakly cartesian transformations between them, and write I for the inclusion. Lemma(10.3) demonstrates that E is a composite

$$[\mathbb{P}, \mathbf{Set}] \xrightarrow{E^\circ} \mathrm{WcEnd}(\mathbf{Set}) \xrightarrow{I} \mathrm{End}(\mathbf{Set})$$

For $T \in \mathrm{End}(\mathbf{Set})$ and $n \in \mathbb{N}$, denote by $r^\circ(T)(n)$ the set of T-generic morphisms $f : 1 \to T(n)$. Lemma(5.10) guarantees that for weakly cartesian $\alpha : T \Rightarrow T'$,

$$f \longmapsto \alpha_n \circ f$$

are well-defined arrow mappings for the functor

$$\mathrm{WcEnd}(\mathbf{Set}) \xrightarrow{r^\circ} [\mathbb{P}, \mathbf{Set}]$$

Recall $r : \mathrm{End}(\mathbf{Set}) \to [\mathbb{P}, \mathbf{Set}]$ from section (3), the functor that corresponds to restriction along $E_{\mathcal{C}} : \mathbb{P} \to \mathbf{Set}$ which regards permutations as bijective functions. Since $r(T)(n)$ may be viewed as the set of all morphisms $1 \to T(n)$, there are natural inclusions which assemble together to provide a monomorphism $\psi : r^\circ \Rightarrow rI$.

10.5. LEMMA. *Let X be a species. Then an element of $rE(X)(n)$ is in the image of $\eta_{X,n}$ iff it is $E(X)$-generic as an arrow $1 \to E(X)(n)$.*

Proof. ($\Rightarrow$): Recalling the definition of $E(X)$, the commutativity of

$$\begin{tikzcd} 1 \arrow[r, "{[x',f]}"] \arrow[d, "{[x,1_n]}"'] & E(X)(A) \arrow[d, "E(X)(\gamma)"] \\ E(X)(n) \arrow[r, "E(X)(\beta)"'] & E(X)(B) \end{tikzcd}$$

where $x \in X(n)$, $x' \in X(m)$, and $f \in A^m$, means that $[x, \beta] = [x', \gamma f]$. That is, $m = n$ and there is $\rho \in \mathrm{Sym}_n$, so that $x = X(\rho)(x')$ and $\beta = \gamma f \rho$. However $[x', f] = [X(\rho)(x'), f\rho]$ by definition, so that without loss of generality we may assume that x' and f were chosen so that $x = x'$ and $\beta = \gamma f$. Thus f is a $E(X)$-fill for this square.
($\Leftarrow$): Let $[x, f] : 1 \to E(X)(n)$ where $f : m \to n$, be $E(X)$-generic. Since $[x, f]$ factors as

$$1 \xrightarrow{[x,1]} E(X)(m) \xrightarrow{E(X)(f)} E(X)(n)$$

and $[x, 1]$ is $E(X)$-generic also, $m = n$ and f is an automorphism of n by lemma(5.7), so that $[x, f] = [X(f^{-1})(x), 1]$. ∎

A conceptual restatement of this lemma is

10.6. Corollary. *η factors as*

$$\begin{tikzcd} {[\mathbb{P}, \mathbf{Set}]} \arrow[r, "1"] \arrow[d, "E^\circ"'] & {[\mathbb{P}, \mathbf{Set}]} \\ \mathrm{WcEnd}(\mathbf{Set}) \arrow[r, "I"'] \arrow[ur, phantom, "\Downarrow\eta"] & \mathrm{End}(\mathbf{Set}) \arrow[u, "r"'] \end{tikzcd} \quad = \quad \begin{tikzcd} {[\mathbb{P}, \mathbf{Set}]} \arrow[r, "1"] \arrow[d, "E^\circ"'] \arrow[dr, phantom, "\Downarrow\eta^\circ", pos=0.25] & {[\mathbb{P}, \mathbf{Set}]} \\ \mathrm{WcEnd}(\mathbf{Set}) \arrow[r, "I"'] \arrow[ur, "r^\circ"] \arrow[ur, phantom, "\Downarrow\psi", pos=0.75] & \mathrm{End}(\mathbf{Set}) \arrow[u, "r"'] \end{tikzcd} \qquad (7)$$

where η° is an isomorphism.

Noting that for $T \in \mathrm{End}(\mathbf{Set})$ and $Z \in \mathbf{Set}$, $E^\circ r^\circ(T)(Z)$ is the subset of $Er(T)(Z)$ consisting of those $[f, g]$ such that $f : 1 \to Tn$ is T-generic, and we define $\varepsilon^\circ : E^\circ r^\circ \Rightarrow 1$ by $\varepsilon^\circ[f, g] = T(g) \circ f$ which by definition satisfies

$$\begin{tikzcd} \mathrm{WcEnd}(\mathbf{Set}) \arrow[r, "r^\circ"] \arrow[d, "I"'] \arrow[dr, phantom, "\Rightarrow\psi", pos=0.3] & {[\mathbb{P}, \mathbf{Set}]} \arrow[d, "E^\circ"] \\ \mathrm{End}(\mathbf{Set}) \arrow[ur, "r"] \arrow[d, "1"'] \arrow[r, phantom, "\Lleftarrow\varepsilon"] & \mathrm{WcEnd}(\mathbf{Set}) \arrow[dl, "I"] \\ \mathrm{End}(\mathbf{Set}) & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathrm{WcEnd}(\mathbf{Set}) \arrow[r, "r^\circ"] \arrow[d, "I"'] \arrow[dr, "1"'] \arrow[dr, phantom, "\Lleftarrow\varepsilon^\circ", pos=0.3, shift left=3] & {[\mathbb{P}, \mathbf{Set}]} \arrow[d, "E^\circ"] \\ \mathrm{End}(\mathbf{Set}) \arrow[d, "1"'] \arrow[r, phantom, "="] & \mathrm{WcEnd}(\mathbf{Set}) \arrow[dl, "I"] \\ \mathrm{End}(\mathbf{Set}) & \end{tikzcd} \qquad (8)$$

10.7. Lemma. *ε° is monic.*

PROOF. Let $f : 1 \to T(n)$ be T-generic, $g : n \to Z$, $f' : 1 \to T(m)$ be T-generic, and $g' : m \to Z$, with $\varepsilon^\circ[f,g] = \varepsilon^\circ[f',g']$. Then

$$\begin{tikzcd} 1 \arrow[r, "f'"] \arrow[d, "f"'] & T(m) \arrow[d, "T(g')"] \\ T(n) \arrow[r, "T(g)"'] & T(Z) \end{tikzcd}$$

commutes, and so has a T-fill that is an isomorphism. That is $m = n$, and there is $\rho \in \mathrm{Sym}_n$ so that $T(\rho)(f) = f'$ and $g'\rho = g$. Thus, $[f,g] = [T(\rho^{-1})(f'), g'\rho] = [f',g']$. ∎

Recall that an arrow f in a 2-category $\mathcal{K}$ is *representably faithful* when for any $K \in \mathcal{K}$, the functor $\mathcal{K}(K,f)$ is faithful.

10.8. LEMMA. *Consider an adjunction η, ε: $IE^\circ \dashv r$, 2-cells $\eta^\circ : 1 \Rightarrow r^\circ E^\circ$ and $\varepsilon^\circ : E^\circ r^\circ \Rightarrow 1$, and, a monomorphism $\psi : r^\circ \Rightarrow rI$, in a 2-category, where I is representably faithful, and suppose that this data satisfies (7) and (8).*
Then η°, ε°: $E^\circ \dashv r^\circ$.

PROOF. The triangular identities for η° and ε° follow immediately from those for η and ε, and the hypotheses of this lemma. ∎

Applying (10.8) to the present situation produces

$$\begin{tikzcd} \mathrm{WcEnd}(\mathbf{Set}) \arrow[r, bend right, "r^\circ"'] \arrow[r, phantom, "\perp"] & {[\mathbb{P}, \mathbf{Set}]} \arrow[l, bend right, "E^\circ"'] \end{tikzcd}$$

with unit η° an isomorphism and counit ε° a monomorphism. Thus we obtain an elementary characterisation of the image of E in terms of generics and weakly cartesian transformations.

10.9. PROPOSITION.

1. *$T \in \mathrm{End}(\mathbf{Set})$ is analytic iff every $1 \to TZ$ factors as*

$$1 \xrightarrow{g} Tn \xrightarrow{T(h)} TZ$$

where g is T-generic and $n \in \mathbb{N}$.

2. *$EX \Rightarrow EY$ is analytic iff it is weakly cartesian.*

PROOF. (1): $T \in \mathrm{End}(\mathbf{Set})$ is analytic iff ε°_T is epic, which amounts to the condition that every $1 \to TZ$ factors as above.
(2): E° is fully-faithful since η° is an isomorphism. ∎

This result appears as propositions 2 and 3 of the appendix of [Joy86].

10.10. Theorem. *For a finitary endofunctor T of* **Set** *the following statements are equivalent.*

1. *T is analytic.*
2. *T admits generic factorisations.*
3. *There is a unique $\phi : T \Rightarrow \mathcal{C}$ which preserves and reflects generics.*

Moreover when T satisfies these hypotheses, $f : A \to TB$ is generic and A is a finite set $\Rightarrow$ B is finite.

Proof. (1)$\Rightarrow$(2): Consider f as in (2) we factor it by choosing $\overline{f}$ so that $f = c_{T,Z} \circ \overline{f}$ (using the surjectivity of $c_{T,Z}$), and factoring $\overline{f}$ using (10.1)

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\overline{f}"] \arrow[ddr, bend right, "g"'] & & T(Z) \\ & \overline{T}(Z) \arrow[ur, "c_{T,Z}"'] & \\ & \overline{T}(H) \arrow[u, "\overline{T}(h)"'] \arrow[r, "c_{T,H}"'] & T(H) \arrow[uu, "T(h)"'] \end{tikzcd}$$

so that g is $\overline{T}$-generic. By (5.10) and (10.4), $c_{T,H} \circ g$ is T-generic. Moreover, if A is finite and f is generic then B is finite by (10.1).
(1)$\Rightarrow$(3): By (1) there is a species X and a natural isomorphism $\iota : T \Rightarrow E(X)$. Composing with ι gives a bijection $\mathrm{WcEnd}(\mathbf{Set})(T, \mathcal{C}) \cong \mathrm{WcEnd}(\mathbf{Set})(E(X), \mathcal{C})$. Since $E(1) = \mathcal{C}$, $\mathrm{WcEnd}(\mathbf{Set})(E(X), \mathcal{C}) \cong 1$ by (10.9)(2). That is, there is a unique weakly cartesian $\phi : T \Rightarrow \mathcal{C}$, and so it suffices to show that ϕ reflects generics. By (1)$\Rightarrow$(2) T admits generic factorisations, and so by (5.13)(1), ϕ reflects generics.
(3)$\Rightarrow$(2): Since $\mathcal{C}$ admits generic factorisations the result follows from (5.13)(2).
(2)$\Rightarrow$(1): Immediate from (10.9)(1). ∎

10.11. Theorem. *For $\phi : S \Rightarrow T$, a natural transformation between analytic functors, the following statements are equivalent:*

1. *ϕ is analytic.*
2. *ϕ preserves and reflects generics.*
3. *ϕ is weakly cartesian.*

Proof. (1)$\Leftrightarrow$(3) by (10.9). Since S admits generic factorisations by (10.10), (3)$\Leftrightarrow$(2) by (5.10) and (5.13). ∎

By comparing (10.11) and (10.2) the following observation of Batanin in [Bat02a] is immediate.

10.12. Corollary. *Let S and T be strongly analytic and $\phi : S \Rightarrow T$. Then ϕ is weakly cartesian $\Rightarrow$ ϕ is cartesian.*

It is shown in the appendix of [Joy86] that $T \in \mathrm{End}(\mathbf{Set})$ is analytic iff it is finitary, cofinitary and weak pullback preserving. Given the above characterisation of analytic functors one direction of Joyal's result is now immediate. Namely, from (10.10), (5.17) and (6.8) analytic functors possess these properties.

11. Weakly cartesian monads and their operads

We now exhibit the generalised notion of cartesian monad and operad over it as promised in (2.7)(5). First we require a notion of collection.

11.1. Definition. *Let $T \in \mathrm{End}(\mathcal{A})$. The category* T-Coll *of* T-collections *is defined as the full subcategory of* $\mathrm{End}(\mathcal{A}){\downarrow}T$ *consisting of the $\sigma : S \Rightarrow T$ such that S admits generic factorisations and σ is weakly cartesian.*

In general, T-Coll here is different to the definition of T-Coll given in section(2). However by proposition(5.13)(3) the definitions coincide when T admits strict generic factorisations.

11.2. Definition. *A monad (T, η, μ) on $\mathcal{A}$ is* weakly cartesian *when it satisfies the following axioms:*

1. *T admits generic factorisations.*
2. *η and μ are weakly cartesian.*

11.3. Remark. By (5.10), (5.13) and (5.15), "weakly cartesian" in (11.2) and (11.1) can be replaced by "preserves and reflects generics".

An immediate consequence of (5.15) and (5.16) is

11.4. Proposition. *Let T be a weakly cartesian monad on $\mathcal{A}$. Then* T-Coll *is a monoidal subcategory of* $\mathrm{End}(\mathcal{A}){\downarrow}T$.

11.5. Definition. *Let T be a weakly cartesian monad on $\mathcal{A}$.*

1. *A* T-operad *is a monoid in* T-Coll. *Thus, a T-operad consists of a weakly cartesian monad morphism $S \Rightarrow T$.*
2. *An* algebra *of a T-operad $S \Rightarrow T$ is an algebra of the monad S.*

11.6. Examples.

1. Cartesian monads are weakly cartesian monads whose generics are strict. The collections and operads defined here coincide with those defined in section(2).

2. $\mathcal{C}$ (the commutative monoid monad on **Set**) is a weakly cartesian monad. A symmetric operad in **Set** is usually defined to be a monoid in $[\mathbb{P}, \mathbf{Set}]$ for substitution of species. The functor E is strong monoidal with respect to substitution of species and composition of endofunctors. Moreover, it is easy to verify directly that E is faithful, and so by definition, provides a monoidal equivalence between $[\mathbb{P}, \mathbf{Set}]$ and the category of analytic functors and analytic natural transformations. Thus a symmetric operad amounts to a monad T on **Set** whose functor part, unit and multiplication are all analytic. That is, by the results of the previous section symmetric operads in **Set** are precisely $\mathcal{C}$-operads.

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