Foreword

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The early 60s was a great time in America for a young mathematician. Washington had responded to Sputnik with a lot of money for science education and the scientists, bless them, said that they could not do anything until students knew mathematics. What Sputnik proved, incredibly enough, was that the country needed more mathematicians.

Publishers got the message. At annual AMS meetings you could spend entire evenings crawling publishers' cocktail parties. They weren't looking for book buyers, they were looking for writers and somehow they had concluded that the best way to get mathematicians to write elementary texts was to publish their advanced texts. Word had gone out that I was writing a text on something called "category theory" and whatever it was, some big names seemed to be interested. I lost count of the bookmen who visited my office bearing gift copies of their advanced texts. I chose Harper & Row because they promised a low price ($\leq$ \$8) and—even better—hundreds of free copies to mathematicians of my choice. (This was to be their first math publication.)

On the day I arrived at Harper's with the finished manuscript I was introduced, as a matter of courtesy, to the Chief of Production who asked me, as a matter of courtesy, if I had any preferences when it came to fonts and I answered, as a matter of courtesy, with the one name I knew, New Times Roman.

It was not a well-known font in the early 60s; in those days one chose between Pica and Elite when buying a typewriter—not fonts but sizes. The Chief of Production, no longer acting just on courtesy, told me that no one would choose it for something like mathematics: New Times Roman was believed to be maximally dense for a given level of legibility. Mathematics required a more spacious font. All that was news to me; I had learned its name only because it struck me as maximally elegant.

The Chief of Production decided that Harper's new math series could be different. Why not New Times Roman? The book might be even cheaper than \$8 (indeed, it sold for \$7.50). We decided that the title page and headers should be *sans serif* and settled that day on Helvetica (it ended up as a rather non-standard version). Harper & Row became enamored with those particular choices and kept them for the entire series. (And—coincidently or not—so, eventually, did the world of desktop publishing.) The heroic copy editor later succeeded in convincing the Chief of Production that I was right in asking for negative page numbering. The title page came in at a glorious –11 and—best of all—there was a magnificent page 0.

The book's sales surprised us all; a second printing was ordered. (It took us a while to find out who all the extra buyers were: computer scientists.) I insisted on a number of changes (this time Harper's agreed to make them without deducting from my royalties; the correction of my left-right errors—scores of them—for the first printing had cost me hundreds of dollars). But for reasons I never thought to ask about, Harper's didn't mark the second printing as such. The copyright page, –8, is almost identical, even the date. (When I need to determine which printing I'm holding—as, for example, when finding a copy for this third "reprinting"—I check the last verb on page –3. In the second printing it is *has* instead of *have*).

A few other page-specific comments:

Page 8: Yikes! In the first printing there's no definition of natural equivalence. Making room for it required much shortening of this paragraph from the first printing:

> Once the definitions existed it was quickly noticed that functors and natural transformations *had* become a major tool in modern mathematics. In 1952 Eilenberg and Steenrod published their *Foundations of Algebraic Topology* [7], an axiomatic approach to homology theory. A *homology theory* was defined as a functor from a topological category to an algebraic category obeying certain axioms. Among the more striking results was their classification of such "theories," an impossible task without the notion of natural equivalence of functors. In a fairly explosive manner, functors and natural transformations have permeated a wide variety of subjects. Such monumental works as Cartan and Eilenberg's *Homological Algebra* [4], and Grothendieck's *Elements of Algebraic Geometry* [1] testify to the fact that functors have become an established concept in mathematics.

Page 21: The term "difference kernel" in 1.6 was doomed, of course, to be replaced by the word "equalizer".

Pages 29–30: Exercise 1–D would have been much easier if it had been delayed until after the definitions of generator and pushout. The category [→] is best characterized as a generator for the category of small categories that appears as a retract of every other generator. The category [→→] is a pushout of the two maps from 1 to [→] and this characterization also simplifies the material in section 3: if a functor fixes the two maps from 1 to [→] then it will be shown to be equivalent to the identity functor; if, instead, it twists them it is equivalent to the dual-category functor. These characterizations have another advantage: they are correct. If one starts with the the two-element monoid that isn't a group, views it as a category and then formally "splits the idempotents" (as in Exercise 2–B, page 61) the result is another two-object category with exactly three endofunctors. And the supposed characterization of [→→] is counterexampled by the disjoint union of [→] and the cyclic group of order three.

Page 35: The axioms for abelian categories are redundant: either **A 1** or **A 1*** suffices, that is, each in the presence of the other axioms implies the other. The proof, which is not straightforward, can be found on section 1.598 of my book with Andre Scedrov, *Categories, Allegories* [North Holland, 1990], henceforth to be referred to as *Cats & Alligators*. Section 1.597 of that book has an even more parsimonious definition of abelian category (which I needed for the material described below concerning page 108): it suffices to require either products or sums and that every map has a "normal factorization", to wit, a map that appears as a cokernel followed by a map that appears as kernel.

Pages 35–36: Of the examples mentioned to show the independence of **A 3** and **A 3*** one is clear, the other requires work: it is not exactly trivial that epimorphisms in the category of groups (abelian or not) are onto—one needs the "amalgamation lemma". (Given the symmetry of the axioms either one of the examples would, note, have sufficed.) For the independence of **A 2** (hence, by taking its dual, also of **A 2***) let R be a ring, commutative for convenience. The full subcategory, $\mathcal{F}$, of finitely presented R-modules is easily seen to be closed under the formation of cokernels of arbitrary maps—quite enough for **A 2*** and **A 3**. With a little work one can show that the kernel of any epi in $\mathcal{F}$ is finitely generated which guarantees that it is the image of a map in $\mathcal{F}$ and that's enough for **A 3***. The necessary and sufficient condition that $\mathcal{F}$

satisfy **A 2** is that R be "coherent", that is, all of its finitely generated ideals be finitely presented as modules. For present purposes we don't need the necessary and sufficient condition. So: let K be a field and R be the result of adjoining a sequence of elements X_n subject to the condition that $X_iX_j = 0$ all i, j. Then multiplication by, say, X_1 defines an endomorphism on R, the kernel of which is not finitely generated. More to the point, it fails to have a kernel in $\mathcal{F}$.

Page 60: Exercise 2–A on additive categories was entirely redone for the second printing. Among the problems in the first printing were the word "monoidal" in place of "pre-additive" (clashing with the modern sense of monoidal category) and—would you believe it!—the absence of the distributive law.

Page 72: A reviewer mentioned as an example of one of my private jokes the size of the font for the title of section 3.6, BIFUNCTORS. Good heavens. I was not really aware of how many jokes (private or otherwise) had accumulated in the text; I must have been aware of each one of them in its time but I kept no track of their number. So now people were seeking the meaning for the barely visible slight increase in the size of the word BIFUNCTORS on page 72. If the truth be told, it was from the first sample page the Chief of Production had sent me for approval. Somewhere between then and when the rest of the pages were done the size changed. But BIFUNCTORS didn't change. At least not in the first printing. Alas, the joke was removed in the second printing.

Pages 75–77: Note, first, that a root is defined in Exercise 3–B not as an object but as a constant functor. There was a month or two in my life when I had come up with the notion of reflective subcategories but had not heard about adjoint functors and that was just enough time to write an undergraduate honors thesis [Brown University, 1958]. By constructing roots as coreflections into the categories of constant functors I had been able to prove the equivalence of completeness and co-completeness (modulo, as I then wrote, "a set-theoretic condition that arises in the proof"). The term "limit" was doomed, of course, not to be replaced by "root". Saunders Mac Lane predicted such in his (quite favorable) review, thereby guaranteeing it. (The reasons I give on page 77 do not include the really important one: I could not for the life of me figure out how $A \times B$ results from a limiting process applied to A and B. I still can't.)

Page 81: Again yikes! The definition of representable functors in Exercise 4–G appears only parenthetically in the first printing. When rewritten to give them their due it was necessary to remove the sentence "To find A, simply evaluate the left-adjoint of S on a set with a single element." The resulting paragraph is a line shorter; hence the extra space in the second printing.

Page 84: After I learned about adjoint functors the main theorems of my honors thesis mutated into a chapter about the general adjoint functor theorems in my Ph.D. dissertation [Princeton, 1960]. I was still thinking, though, in terms of reflective subcategories and still defined the limit (or, if you insist, the root) of $\mathcal{D} \to \mathcal{A}$ as its reflection in the subcategory of constant functors. If I had really converted to adjoint functors I would have known that limits of functors in $\mathcal{A}^{\mathcal{D}}$ should be defined via the right adjoint of the functor $\mathcal{A} \to \mathcal{A}^{\mathcal{D}}$ that delivers constant functors. Alas, I had not totally converted and I stuck to my old definition in Exercise 4–J. Even if we allow that the category of constant functors can be identified with $\mathcal{A}$ we're in trouble when $\mathcal{D}$ is empty: no empty limits. Hence the peculiar "condition zero" in the statement of the general adjoint functor theorem and any number of requirements to come about zero objects and such, all of which are redundant when one uses the right definition of limit.

There is one generalization of the general adjoint functor theorem worth mentioning here. Let "weak-" be the operator on definitions that removes uniqueness conditions. It suffices that all small diagrams in $\mathcal{A}$ have weak limits and that T preserves them. See section 1.8 of *Cats & Alligators*. (The weakly complete categories of particular interest are in homotopy theory. A more categorical example is COSCANECOF, the category of small categories and natural equivalence classes of functors.)

Pages 85–86: Only once in my life have I decided to refrain from further argument about a non-baroque matter in mathematics and that was shortly after the book's publication: I refused to engage in the myriad discussions about the issues discussed in the material that starts on the bottom of page 85. It was a good rule. I had (correctly) predicted that the controversy would evaporate and that, in the meantime, it would be a waste of time to amplify what I had already written. I should, though, have figured out a way to point out that the forgetful functor for the category, $\mathcal{B}$, described on pages 131–132 has all the conditions needed for the general adjoint functor except for the solution set condition. Ironically there was already in hand a much better example: the forgetful functor from the category of complete boolean algebras (and bi-continuous homomorphisms) to the category of sets does not have a left adjoint (put another way, free complete boolean algebras are non-existently large). The proof (albeit for a different assertion) was in Haim Gaifman's 1962 dissertation [Infinite Boolean Polynomials *I. Fund. Math.* 54 1964].

Page 87: The term "co-well-powered" should, of course, be "well-co-powered".

Pages 91–93: I lost track of the many special cases of Exercise 3–O on model theory that have appeared in print (most often in proofs that a particular category, for example the category of semigroups, is well-co-powered and in proofs that a particular category, for example the category of small skeletal categories, is co-complete). In this exercise the most conspicuous omission resulted from my not taking the trouble to allow many-sorted theories, which meant that I was not able to mention the easy theorem that $\mathcal{B}^{\mathcal{A}}$ is a category of models whenever $\mathcal{A}$ is small and $\mathcal{B}$ is itself a category of models.

Page 107: Characteristic zero is not needed in the first half of Exercise 4–H. It would be better to say that a field arising as the ring of endomorphisms of an abelian group is necessarily a prime field (hence the category of vector spaces over any non-prime field can not be fully embedded in the category of abelian groups). The only reason I can think of for insisting on characteristic zero is that the proofs for finite and infinite characteristics are different—a strange reason given that neither proof is present.

Page 108: I came across a good example of a locally small abelian category that is not very abelian shortly after the second printing appeared: to wit, the target of the universal homology theory on the category of connected CW-complexes (finite dimensional, if you wish). Joel Cohen called it the "Freyd category" in his book Stable Homotopy [*Lecture Notes in Mathematics* Vol. 165 Springer-Verlag, Berlin-New York 1970], but it should be noted that Joel didn't name it after me. (He always insisted that it was my daughter.) It's such a nice category it's worth describing here. To construct it, start with pairs of CW-complexes $\langle X', X\rangle$ where X' is a non-empty subcomplex of X and take the obvious condition on maps, to wit, $f : \langle X', X\rangle \to \langle Y', Y\rangle$ is a continuous map $f : X \to Y$ such that $f(X') \subseteq Y'$. Now impose the congruence that identifies $f, g : \langle X', X\rangle \to \langle Y', Y\rangle$ when $f|X'$ and $g|X'$ are homotopic (as maps to Y). Finally, take the result of formally making the suspension functor an automorphism (which can, of course, be restated as taking a reflection). This can all be found in Joel's book or in my article with the same title as Joel's, Stable Homotopy, [*Proc. of the Conference of Categorical Algebra*, Springer-Verlag, 1966]. The fact that it is not very abelian follows from the fact that the stable-homotopy category appears as a subcategory (to wit, the full subcategory of objects of the form $\langle X, X\rangle$) and that category was shown not to have any embedding at all into the category of sets in Homotopy Is Not Concrete, [*The Steenrod Algebra and its Applications*, Lecture Notes in

Mathematics, Vol. 168 Springer, Berlin 1970]. I was surprised, when reading page 108 for this Foreword, to see how similar in spirit its set-up is to the one I used 5 years later to demonstrate the impossibility of an embedding of the homotopy category.

Page (108): Parenthetically I wrote in Exercise 4–I, “The only [non-trivial] embedding theorem for large abelian categories that we know of [requires] both a generator and a cogenerator.” It took close to ten more years to find the right theorem: an abelian category is very abelian iff it is well powered (which it should be noticed, follows from there being any embedding at all into the category of sets, indeed, all one needs is a functor that distinguishes zero maps from non-zero maps). See my paper Concreteness [*J. of Pure and Applied Algebra*, Vol. 3, 1973]. The proof is painful.

Pages 118–119: The material in small print (squeezed in when the first printing was ready for bed) was, sad to relate, directly disbelieved. The proofs whose existence are being asserted are natural extensions of the arguments in Exercise 3–O on model theory (pages 91–93) as suggested by the “conspicuous omission” mentioned above. One needs to tailor Lowenheim-Skolem to allow first-order theories with infinite sentences. But it is my experience that anyone who is conversant in both model theory and the adjoint-functor theorems will, with minimal prodding, come up with the proofs.

Pages 130–131: The Third Proof in the first printing was hopelessly inadequate (and Saunders, bless him, noticed that fact in his review). The proof that replaced it for the second printing is OK. Fitting it into the alloted space was, if I may say so, a masterly example of compression.

Pages 131–132: The very large category $\mathcal{B}$ (Exercise 6–A)—with a few variations—has been a great source of counterexamples over the years. As pointed out above (concerning pages 85–86) the forgetful functor is bicontinuous but does not have either adjoint. To move into a more general setting, drop the condition that G be a group and rewrite the “convention” to become $f(y) = 1_G$ for $y \notin S$ (and, of course, drop the condition that $h : G \to G'$ be a homomorphism—it can be any function). The result is a category that satisfies all the conditions of a Grothendieck topos except for the existence of a generating set. It is not a topos: the subobject classifier, Ω, would need to be the size of the universe. If we require, instead, that all the values of all $f : S \to (G, G)$ be permutations, it is a topos and a boolean one at that. Indeed, the forgetful functor preserves all the relevant structure (in particular, Ω has just two elements). In its category of abelian-group objects—just as in $\mathcal{B}$—$\mathrm{Ext}(A, B)$ is a proper class iff there's a non-zero group homomorphism from A to B (it needn't respect the actions), hence the only injective object is the zero object (which settled a once-open problem about whether there are enough injectives in the category of abelian groups in every elementary topos with natural-numbers object.)

Pages 153–154: I have no idea why in Exercise 7–G I didn't cite its origins: my paper, Relative Homological Algebra Made Absolute, [*Proc. Nat. Acad. Sci.*, Feb. 1963].

Page 158: I must confess that I cringe when I see “A man learns to think categorically, he works out a few definitions, perhaps a theorem, more likely a lemma, and then he publishes it.” I cringe when I recall that when I got my degree, Princeton had never allowed a female student (graduate or undergraduate). On the other hand, I don't cringe at the pronoun “he”.

Page 159: The Yoneda lemma turns out not to be in Yoneda's paper. When, some time after both printings of the book appeared, this was brought to my (much chagrined) attention, I brought it the attention of the person who had told me that it was the Yoneda lemma. He consulted his notes and discovered that it appeared in a lecture that Mac Lane gave on Yoneda's treatment of the higher *Ext* functors. The name “Yoneda lemma” was not doomed to be replaced.

Pages 163–164: *Allows* and *Generating* were missing in the index of the first printing as was page 129 for *Mitchell*. Still missing in the second printing are *Natural equivalence, 8* and *Pre-additive category, 60*. Not missing, alas, is *Monoidal category*.

FINALLY, a comment on what I "hoped to be a geodesic course" to the full embedding theorem (mentioned on page 10). I think the hope was justified for the full embedding theorem, but if one settles for the exact embedding theorem then the geodesic course omitted an important development. By broadening the problem to regular categories one can find a choice-free theorem which—aside from its wider applicability in a topos-theoretic setting—has the advantage of naturality. The proof requires constructions in the broader context but if one applies the general construction to the special case of abelian categories, we obtain:

There is a construction that assigns to each small abelian category $\mathbb{A}$ *an exact embedding into the category of abelian groups* $\mathbb{A} \to \mathcal{G}$ *such that for any exact functor* $\mathbb{A} \to \mathbb{B}$ *there is a natural assignment of a natural transformation from* $\mathbb{A} \to \mathcal{G}$ *to* $\mathbb{A} \to \mathbb{B} \to \mathcal{G}$*. When* $\mathbb{A} \to \mathbb{B}$ *is an embedding then so is the transformation.*

The proof is suggested in my pamphlet *On canonizing category theory or on functorializing model theory* [mimeographed notes, Univ. Pennsylvania, Philadelphia, Pa., 1974] It uses the strange subject of τ-categories. More accessibly, it is exposed in section 1.54 of *Cats & Alligators.*

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