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THE TOPOS OF BALL COMPLEXES

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Foreword by F.W. Lawvere

There are hints in the 1983 Pursuing Stacks of Alexander Grothendieck, that parallel to the simplicial core of piecewise linear geometry, another topos should exist expressing the core of quadratic geometry. Mike Roy successfully undertook the construction of such a topos. It is based on the intuition that spheres can be defined in terms of balls in two opposite ways. (The boundary of a 3-ball is the union of two copies (hemispheres) of the 2-ball, and the equator of those two embeddings is itself the union of two copies of a 1-ball. Similar combinatorics persist into higher dimensions.) Investigating these relations suggested the definition of a 'non-linear homology' whose possible applications still need to be further explored. The thesis also provided one of the crucial inputs to later determination of Aufhebung relations, and in particular to the qualitative description of dimension 1.

Editor's note: Page 55 of the thesis was removed from this reprint (it only contained a single typo). Other than that, the version presented here is as in the original thesis.

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Abstract

We make a detailed investigation of the topos of Ball Complexes $\mathcal{E}$. This is a presheaf topos whose site $\mathbf{B}$ is the category with objects B_n, the n-balls, and for successive balls , the inclusion of the upper and lower hemisphere and a common retraction squashing the ball onto its solid equator $B_n \;\substack{\xhookrightarrow{\delta_0} \\ \xleftarrow{p} \\ \xhookrightarrow[\delta_1]{}}\; B_{n+1}$. For any three successive balls $B_n \xhookrightarrow{\delta_i} B_{n+1} \underset{\delta_k}{\overset{\delta_j}{\rightrightarrows}} B_{n+2}$ we have $\delta_j \delta_i = \delta_k \delta_i$ and no further relations.

We demonstrate some of the remarkable properties of $\mathcal{E}$. We compare this topos with two other toposes which are commonly used in combinatorial topology, observing some surprising similarities between all three and also certain striking contrasting features.

Chapter 1

Introduction

In this dissertation we make a study of the topos of Ball Complexes $\mathcal{E}$. This is a presheaf topos whose site $\mathbf{B}$ is the category with objects B_n, the n-balls, and for successive balls , the inclusion of the upper and lower hemisphere and a common retraction squashing the ball onto its solid equator $B_n \overset{\xhookrightarrow{\delta_0}}{\underset{\xhookrightarrow[\delta_1]{}}{\leftarrow p -}} B_{n+1}$. For any three successive balls $B_n \xhookrightarrow{\delta_i} B_{n+1} \overset{\xhookrightarrow{\delta_j}}{\underset{\delta_k}{\hookrightarrow}} B_{n+2}$ we have $\delta_j\delta_i = \delta_k\delta_i$ and no further relations. Hence any map in $\mathbf{B}$ can be factored as a (split) epimorphism followed by a (split) monomorphism.

The first object of $\mathcal{E}$ that we study is the n-sphere S^n. Intuitively, a sphere can be obtained from gluing two hemispheres, but also a sphere is that part where certain upper and lower hemispheres agree. Hence we show that S^n can be obtained as a pushout and an equalizer.

The topos of Ball Complexes is shown to satisfy the axioms for a "topos of spaces" (see [4]). Two other important examples of toposes of spaces are mentioned in this paper, namely simplicial sets and the Boolean algebra classifier. Throughout this dissertation these topoi are often compared with $\mathcal{E}$. For example, in all three cases one can truncate the site obtaining a full subcategory and thus, on taking Kan extensions, an essential subtopos or level in the ambient topos. Composing these functors we obtain skeletal and coskeletal endofunctors. In all these examples the double negation sheaves are just sets and constitute level zero. The smallest level for which the components of any $X \in \mathcal{E}$ equals the components of the skeleton of X is called level 1. (Of course we have this characterization for any topos equipped with a connected components functor to level zero). In $\mathcal{E}$, level 1 is the truncation at $\mathbf{B}_1$, the topos of reflexive graphs.This is also true of simplicial sets; it is also true that in the Boolean algebra classifier level 1 is the truncation at the second object in the site, but here this is a different topos of graphs called two-way reflexive graphs. An equivalent characterization of level 1 is that it is the smallest level for which the coskeletal inclusion preserves coproducts. In fact, these topoi have the property that not only level 1 but all levels greater than level 0 have their sheaf inclusion preserving coproducts.

For any presheaf topos one is always interested in finding the smallest level that generates by colimits. Lawvere has shown that reflexive graphs generate simplicial sets and that the Boolean algebra classifier is codiscretely generated. By "generate" we mean that an object of simplicial sets is a quotient of a sum of 1-coskeletal objects, and an object of the Boolean algebra classifier is a quotient of a sum of codiscrete objects. Moreover, in either topos every object is a quotient of a 1-coskeletal object.We show that for presheaf topoi of this ilk, a "level n generates" is equivalent to "$cosk_n(C) = C$" for every representable C. We then prove that no level can generate $\mathcal{E}$ by showing that, for any n we can find an m for which $cosk_n(B_m) \neq B_m$, where $B_m \in \mathbf{B}$. Another sharp distinction between $\mathcal{E}$ and simplicial sets or the Boolean algebra classifier is that each subcategory $\mathbf{B}_n \hookrightarrow \mathbf{B}$ has a left adjoint retraction, something not true of the other sites. Thus at the level of essential toposes there is a further functor π_n, left adjoint to the skeletal inclusion. This functor is in a sense like π_0 in that they are both obtained as reflexive coequalizers; in particular, all the π_n functors preserve finite products.

For a given topos $\mathcal{X}$ and level $\mathcal{X}_n$ in $\mathcal{X}$, one can ask if there is a smallest level $\mathcal{X}_m$ for which the n-skeletal inclusion of $\mathcal{X}_n$ in $\mathcal{X}$ factors through the m-coskeletal inclusion of $\mathcal{X}_m$ i n $\mathcal{X}$. Lawvere (see [5], [6], *and* [9]) has given the name "Aufhebung of level n" to the level m (if it exists). In $\mathcal{E}$, the Aufhebung function is simply the successor function; that is, level n = Aufhebung of level n-1 = n^{th} truncated topos. Furthermore, one can define coAufhebung in the obvious way and this suprisingly exists for Ball Complexes and is also the successor function; that is, if X in $\mathcal{E}$ is n-coskeletal, then X is $(n+1)-skeletal.$

In [4], Lawvere showed that if X is a reflexive graph, the subcategory $S(X)$ of $\mathcal{S}^{\mathbf{B}_1^{op}}/X$ consisting of $Y \to X$ discretely fibered over X is a topos, and furthermore, if X is replaced by the loop $L =$ , then $S(L)$ is the topos of irreflexive graphs. A similar result is true for $\mathcal{E}$: Using an appropriate $L \in \mathcal{E}$ and the same construction, one obtains $S(L) = \mathcal{S}^{\mathbf{B}_{mono}^{op}}$, where $\mathbf{B}_{mono}$ is the subcategory of $\mathbf{B}$ consisting only of monomorphisms. In the lower hemisphere(Australia) this is known as the topos of globular sets.

Another feature that $\mathcal{E}$ and simplicial sets have in common is the Dold-Kan-Moore Theorem, which states that the category of abelian objects in simplicial sets is equivalent to the category of chain complexes. The result is also true for Ball Complexes, but here the proof is much simpler than in the simplicial case. However, for an abelian object in simplicial sets , the homology groups of the associated chain complex and "Moore complex" are isomorphic, but in the case of Ball Complexes, with boundary given by taking the difference of hemispheres in some arbitrarily chosen manner, these two complexes are not isomorphic even if the abelian object is freely generated by a ball complex of sets.

Since the topos of Ball Complexes comes equipped with π_n functors that preserve finite products, we are able to define homotopy categories $\mathcal{H}_n(\mathcal{E})$ for each $n \in \mathbf{N}$. The objects of these categories are those of $\mathcal{E}$ and the hom-sets are $\mathcal{H}_n(\mathcal{E})(X,Y) = \pi_n(Y^X)(B_n)$ for $X, Y \in \mathcal{E}$. However, for $n > 0$, the only terminal objects in $\mathcal{H}_n(\mathcal{E})$ are already terminal in $\mathcal{E}$.For n=0, this is not the case. We show that the representables B_n are "contractible" in $\mathcal{E}$ in the sense that they are terminal objects of $\mathcal{H}_0(\mathcal{E})$.Furthermore, we prove the surprising result that S^n is contractible.

In the last section, we consider the points of $\mathcal{E}$. The filtering functors $\mathbf{B} \to \mathcal{S}$ in some sense correspond to those subsets of $\mathbf{N}$ containing 0. The corresponding points have inverse image functors, which when applied to a ball complex, take the sum of all the nondegenerates at stages labelled by the subset.

Chapter 2

Ball Complexes

2.1 The site: The category of balls B

The category of balls **B** is the category with objects $B_n, n \in \mathbf{N}$ to be thought of as solid $n - balls$, and for any two successive objects B_n, B_{n+1}, two inclusions δ_0, δ_1 of B_n into B_{n+1}, which we picture as the inclusions of the hemispheres, and a common retraction p for δ_0 and δ_1. For any three successive balls

$$\begin{tikzcd} B_n \arrow[r, hook, shift left=2, "\delta_0"] \arrow[r, hook, shift right=2, "\delta_1"'] & B_{n+1} \arrow[l, two heads, "p" description] \arrow[r, hook, shift left=2, "\delta_0"] \arrow[r, hook, shift right=2, "\delta_1"'] & B_{n+2}, \arrow[l, two heads, "p" description] \end{tikzcd}$$

we have $\delta_j \circ \delta_i = \delta_k \circ \delta_i$ and no further relations. This can be motivated by the following result: the functor $\mathbf{B} \longrightarrow \mathcal{S}$, described on objects as $B_n \longmapsto \{x \in \mathbf{R}^n : \|x\| \leq 1\}$, where the hemisphere inclusions are $x \longmapsto (x, \pm\sqrt{1 - \|x\|^2})$, and the retract is $x = (x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{n-1})$, is a faithful functor.

Every map f in **B** can be factored uniquely as (split) epimorphism p followed by a (split) monomorphism δ_i. Schanuel has observed that one can define a "suspension" functor T which "shifts" a map f up one level:

$$T(B_m \xrightarrow{p} B_k \xhookrightarrow{\delta_i} B_s) = B_{m+1} \xrightarrow{p} B_{k+1} \xhookrightarrow{\delta_i} B_{s+1}$$

Lawvere points out that there are endofunctors of $\mathbf{B}$ which interchange hemispheres, identify hemispheres, or leave them unchanged at any chosen stages; that is, $4^{\mathbf{N}} \hookrightarrow End(\mathbf{B})$, and we see that there are continuum many suspension functors.

If a map f has image B_k, we may say that f has rank k. In this way we see that every map of rank k is a $k-fold$ suspension of a unique map of rank 0. For any fixed i, the maps $B_k \xhookrightarrow{\delta_i} B_{k+1}, k \geq 0$, are not the components of a natural transformation $1 \xrightarrow{\delta_i} T$, since, for example, the diagram

$$\begin{tikzcd}
B_0 \arrow[r, hook, "\delta_i"] \arrow[d, hook, "\delta_j"'] & B_1 \arrow[d, hook, "T(\delta_j)"] \\
B_1 \arrow[r, hook, "\delta_i"'] & B_2
\end{tikzcd}$$

does not commute if $\delta_j \neq \delta_i$. Similarly, the maps $B_{k+1} \xrightarrow{p} B_k, k \geq 0$, do not form the components of a natural transformation $T \xrightarrow{p} 1$.

In any category the set of endomaps of a given object form a monoid. If for any B_n in $\mathbf{B}$ we denote by M_n the set of all endomaps of B_n, then for any $x, y \in M_n$, $xyx = xy$. In fact,

$$xy = \begin{cases} x & if\ rank(y) \geq \ rank(x), \\ y & if\ rank(y) < \ rank(x). \end{cases}$$

Monoids M satisfying the equation $xyx = xy \ \ \forall\, x, y \in M$ are called graphic monoids.

In $\mathbf{B}$ the set of all maps of rank $\geq r$ (r $fixed$) forms a (non full) subcategory of $\mathbf{B}$. Its set-theoretic complement is a two-sided ideal of $\mathbf{B}$; hence, if we restrict these two classes of maps to M_n, the endomaps of rank $\geq r$ form a submonoid of M_n and its complement in M_n is a bi-ideal of M. Bi-ideals play an important role in displaying these monoids (see [11]). Between any two objects B_m and B_n there are $2n+1$ maps $B_m \longrightarrow B_n$ if $m \geq n$ and $2m+2$ if $m < n$. Hence in M_n there are $2n+1$ elements, two of which are constants: that is, elements c such that $cx = c \ \ \forall x \in M_n$. These two elements "generate" all non identity elements by suspension "restricted" to M_n.

If we restrict our suspension functor T to M_n, since any $x \in M_n$ is uniquely $x = \delta_k p$ for some p, k, $T(x) = \delta_k p \in M_{n+1}$, defining the submonoid of maps of rank ≥ 1. Hence, T induces an injective monoid homomorphism $T_n : M_n \hookrightarrow M_{n+1}$.

2.2 The topos of presheaves on B: Ball Complexes

The topos of Ball Complexes $\mathcal{E}$ is defined to be the category of presheaves on $\mathbf{B}$, $\mathcal{S}^{\mathbf{B}^{op}}$; that is, it consists of all contravariant set-valued functors $X : \mathbf{B}^{op} \longrightarrow \mathcal{S}$ and natural transformations between them.

In the spirit of the Yoneda lemma, if $X \in \mathcal{S}^{\mathbf{B}^{op}}$ and $x \in X(B_n)$ we think of x as a figure of shape B_n in X: $B_n \xrightarrow{x} X$. In this way we may refer to x as an $n - ball$ of X. This gives precise meaning to the upper and lower hemisphere of x, for these are simply the two composites $B_{n-1} \xhookrightarrow{\delta_i} B_n \xrightarrow{x} X$. We say that x is a degenerate $n - ball$ if x factors across an $m - ball$ y with $m < n$:

$$\begin{tikzcd} B_n \arrow[rr, "x"] \arrow[dr] & & X \\ & B_m \arrow[ur, "y"'] & \end{tikzcd}$$

That every degenerate $n - ball$ x can be so expressed uniquely with y non- degenerate is the content of an Eilenberg-Zilber lemma for Ball Complexes: For any $X \in \mathcal{S}^{\mathbf{B}^{op}}$ and any degenerate $n-ball$ x of X, there is a unique non-degenerate $m-ball$ y of X with $m < n$ such that $x = y \cdot p$. The proof of this result is similar to (but simpler than) the proof of the Eilenberg-Zilber lemma for simplicial sets given in [1].

The functor $T : \mathbf{B} \longrightarrow \mathbf{B}$ induces three endofunctors of $\mathcal{S}^{\mathbf{B}^{op}}$, namely, the restriction T^* along T, and the left and right adjoints of T^*, the left and right Kan extensions along T, $T_!$ and T_*. Note that, in contrast with shifting operators, T^* is faithful.

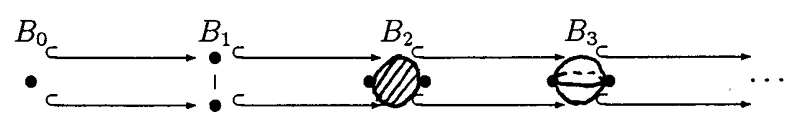
Recall that any presheaf $\mathbf{B}^{op} \xrightarrow{X} \mathcal{S}$ is the colimit $\varinjlim\ (\mathbf{B}/X \xrightarrow{q} \mathbf{B} \xrightarrow{Yoneda} \mathcal{E})$, where q is the labelling functor of the corresponding discrete fibration. For any X in $\mathcal{E}$, the presheaf $T_!(X)$ is given by

$$T_!(X) = \varinjlim_{\mathbf{B}/X}\ (Yoneda\ \circ T \circ q : \mathbf{B}/X \longrightarrow \mathcal{E}).$$

Similarly $T_*(X)$ is an inverse limit.

2.3 Some Exactness Properties

We can picture the first four objects of the category $\mathbf{B}$:



A single point, a closed interval whose two endpoints are the two inclusions of the point, a solid disc with bounding semicircles equal to the two inclusions of the interval, etc. In this picture the union of the two (hemisphere) inclusions $\delta_i : B_{n-1} \hookrightarrow B_n$, the $(n-1)-sphere$ S^{n-1}, is the intersection of two hemisphere inclusions δ_i : $B_n \hookrightarrow B_{n+1}$, the equator of B_{n+1}. For example, S^1 is both the union of the two arcs in B_2 and the equator of B_3. This double description of the spheres persists into the presheaf category as we now show.

Consider the Yoneda embedding $\mathbf{B} \hookrightarrow \mathcal{S}^{\mathbf{B}^{op}}$ of the category $\mathbf{B}$ in presheaves on $\mathbf{B}$. More generally, we would like to show that, $\forall n \geq 0$, the pushout P of the map $2B_{n-2} \to B_{n-1}$ with itself

$$\begin{tikzcd} 2B_{n-2} \arrow[r] \arrow[d] & B_{n-1} \arrow[d, hook, "\beta"] \\ B_{n-1} \arrow[r, hook, "\alpha"'] & P \end{tikzcd}$$

and the pullback Q of δ_0,δ_1 in $\mathcal{E}$

$$\begin{tikzcd} Q \arrow[r, hook, "\mu"] \arrow[d, hook, "\gamma"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_n \arrow[d, hook, "\delta_1"] \\ B_n \arrow[r, hook, "\delta_0"'] & B_{n+1} \end{tikzcd}$$

are the same. (Note that, because of the common retraction p for δ_0 and δ_1, Q is also the equalizer of δ_0, δ_1 and $\gamma = \mu$ is the equalizer inclusion. Also, we may test the pushout P with the object B_{n-1} and identity maps; hence, α and β have a common retraction). Here, $B_{-2} = B_{-1} = \emptyset$, the initial object in $\mathcal{E}$. We now define the (n-1)-sphere in $\mathcal{E}$, S^{n-1}, and show that $P = Q = S^{n-1}$. The (n-1)-sphere is the subobject of B_n whose value at B_k is $S^{n-1}(B_k) = \mathcal{E}_{rank\leq(n-1)}(B_k, B_n)$. This defines a sieve on B_n and hence $S^{n-1} \in \mathcal{E}$. Moreover, the equalizer $Q \hookrightarrow B_n \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} B_{n+1}$ is the sieve on B_n whose elements at stage B_k are maps $B_k \xrightarrow{f} B_n$ for which $\delta_0 \circ f = \delta_1 \circ f$; such an f must factor through an inclusion $B_{n-1} \hookrightarrow B_n$. Hence Q is the sieve generated by $B_{n-1} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} B_n$ and therefore $Q = S^{n-1}$. In particular, there are two inclusions $B_{n-1} \xhookrightarrow{\delta_i} S^{n-1}$ and the diagram

$$\begin{tikzcd}
2B_{n-2} \arrow[r] \arrow[d] & B_{n-1} \arrow[d, hook] \\
B_{n-1} \arrow[r, hook, "\delta_0"'] & S^{n-1}
\end{tikzcd}$$

commutes. If T is any object in $\mathcal{E}$ with $(n-1)-$ *balls* x_0, x_1 such that the outer square in the diagram commutes, i.e., $x_0 \cdot \delta_j = x_1 \cdot \delta_j$, $j \in \{0,1\}$, then there is a unique map θ with $\theta(\delta_i) = x_i$, $i \in \{0,1\}$.

$$\begin{tikzcd}
2B_{n-2} \arrow[r] \arrow[d] & B_{n-1} \arrow[d, hook, "\delta_1"] \arrow[ddr, "x_1"] & \\
B_{n-1} \arrow[r, hook, "\delta_0"'] \arrow[drr, "x_0"'] & S^{n-1} \arrow[dr, "\theta"] & \\
 & & T
\end{tikzcd}$$

(Later, we shall see that S^{n-1} is the "$(n-1)-$ *skeleton*" of B_n, $n \geq 1$).
Observe that $S^0 = 2$ and $S^{-1} = \emptyset$, and that, for $n > 1$, S^{n-1} is a connected subobject of B_n in $\mathcal{E}$. Moreover, if we apply $T_!$ to S^{n-1} (using the pushout form of S^{n-1}) we see that $T_!(S^{n-1}) = S^n$, $n > 1$. Furthermore, since $\delta_i \circ \delta_k = \delta_k$, we see that in the diagram below the composite $\delta_k \cdot i$ equalizes δ_0 and δ_1 and there is, therefore, a unique

induced inclusion of spheres independent of the choice of k

$$\begin{tikzcd}
S^{n-1} \arrow[r, hook, "i"] \arrow[d, dashed] & B_n \arrow[r, hook, shift left] \arrow[r, hook, shift right] \arrow[d, hook, "\delta_k"] & B_{n+1} \\
S^n \arrow[r, hook] & B_{n+1} \arrow[r, hook, shift left, "\delta_0"] \arrow[r, hook, shift right, "\delta_1"'] & B_{n+2}
\end{tikzcd}$$

that is, $S^{n-1} \subset S^n$.

2.4 $\mathcal{E}$ is a Topos of Spaces

In this section we shall show that $\mathcal{E}$ is a "topos of spaces". Recall from [4] that a topos $\mathbf{E}$ defined over another topos $\mathbf{S}$ should at least satisfy the following three axioms to qualify as a topos of spaces.

Axiom 0. $\mathbf{E} \xrightarrow{\Gamma} \mathbf{S}$ is local; i.e, we have (not only) $\Gamma^* \dashv \Gamma_*$, (but also) $\Gamma_* \dashv \Gamma^!$.

$\mathcal{E}$ is defined over the topos of abstract sets, as are all Grothendieck topoi.[1] For any space $X \in \mathcal{E}$, $\Gamma_*(X) = \mathcal{E}(1, X)$, the "points" of X, since 1 is the terminal object of $\mathcal{E}$. Since 1 is representable as $\mathbf{B}(-, B_0)$ we have $\Gamma_* X = X(B_0)$. We may think of (the fully faithful) Γ^* as including the discrete spaces into $\mathcal{E}$ and (the fully faithful) $\Gamma^!$ including the codiscrete/chaotic spaces. For presheaf toposes like $\mathcal{E}$, the values that Γ^* takes are constant presheaves: For any $S \in \mathcal{S}$, $B_n \in \mathbf{B}$, $\Gamma^*(S)(B_n) = S$ and for any $B_n \xrightarrow{f} B_m$, $\Gamma^*(S)(B_m) \xrightarrow{\Gamma^*(S)(f)} \Gamma^*(S)(B_n)$ is the identity map on S. In contrast,

$$\Gamma^!(S)(B_n) = S^{\mathbf{B}(1, B_n)} = \begin{cases} S & if\ n = 0, \\ S^2 & if\ n > 0, \end{cases}$$

[1] $\mathbf{S}$ need not be the category of abstract constant sets, but for simplicity we will assume it is.

and for any $B_n \xrightarrow{f} B_m$, $\mathbf{B}(1, B_m) \xrightarrow{\theta} S$, and $1 \xrightarrow{x} B_n$ in $\mathbf{B}$, $\Gamma^!(S)(f)(\theta)(x) = \theta(fx)$. We may picture $\Gamma^!(S)$ for any $S \in \mathcal{S}$ as $S \rightleftarrows S^2 \xhookrightarrow{\cong} S^2 \cong \ldots$, where the top and bottom arrows are the projections and the middle is the diagonal map. At all other stages, the induced inclusion is an isomorphism. Note also that Γ_* is a retraction of both Γ^* and $\Gamma^!$ ($\Gamma^*, \Gamma^!$ being full and faithful).

The categories of discrete and codiscrete spaces are identical since they are equivalent to $\mathcal{S}$, yet as subcategories of $\mathcal{E}$ they are totally opposite in the following sense: a discrete space totally lacks the cohesion to connect any of its points, whereas, in stark contrast, a codiscrete space has total cohesion: any two points can be connected. Moreover, there is a canonical inclusion $\Gamma^*(S) \hookrightarrow \Gamma^!(S)$ for any $S \in \mathcal{S}$ and these spaces have exactly the same points (apply the points functor Γ_* and this map becomes an isomorphism); but the maps from a codiscrete to a discrete are constant.

Axiom 1. There is a $\Gamma_! \dashv \Gamma^*$ and it preserves finite products

$$\begin{aligned} \Gamma_!(X \times Y) &\simeq \Gamma_!(X) \times \Gamma_!(Y) \\ \Gamma_!(1) &\simeq 1 \qquad\qquad \forall\, X, Y \in \mathcal{E} \end{aligned}$$

$\Gamma_!$ is called the connected components functor and for any X in $\mathcal{E}$, $\Gamma_!(X)$ is its discrete space of components. In the case that $\mathbf{E}$ is the topos of presheaves on a small category $\mathbf{C}$,

$\Gamma_!$ always exists and its value at any X in $\mathbf{E}$ is $\Gamma_!(X) = \varinjlim_{C \in \mathbf{C}} X(C)$. However, $\Gamma_!$ can often fail to preserve finite products. For example, for a generalized space such as right $G-sets$, G a group, $\Gamma_!(X)$ is the set of orbits of the right $G-set\ \ X$. Taking G to be the right regular representation of G (a $G-set$), we have $\Gamma_!(G) = 1$, but $\Gamma_!(G \times G) = |G|$ since each orbit of $G \times G$ is generated by $< g, 1 >$ for $g \in G$.

The importance of the product preservation is that it enables us to construct a homotopy category from $\mathbf{E}$. First note that, like any cartesian closed category, $\mathbf{E}$ is enriched in itself with homs $Y^X \quad \forall\ X, Y$ in $\mathbf{E}$. This makes sense since there is a canonical composition map $Z^Y \times Y^X \longrightarrow Z^X$ for any X, Y, Z in $\mathbf{E}$.

If $\mathbf{A}$ is any category that is enriched in $\mathbf{E}$ it has of course an underlying category $\overline{\mathbf{A}}$ enriched in $\mathcal{S}$ whose homs can be defined by $\overline{\mathbf{A}}(X,Y) = \mathbf{E}(1, A(X,Y))$, $X, Y \in \mathbf{A}$; in case $\mathbf{A} = \mathbf{E}$ this recovers $\mathbf{E}$. But we can also define a different category $[\mathbf{A}]$ that is also enriched in $\mathcal{S}$: Again $[\mathbf{A}]$ has as objects those of $\mathbf{A}$ but we now take $[\mathbf{A}](X,Y) = \Gamma_!(\mathbf{A}(X,Y))$; it is essential that $\Gamma_!$ preserve products for composition to be defined. Specializing to $\mathbf{A} = \mathbf{E}$, we have a homotopy category $[\mathbf{E}]$ with objects those of $\mathbf{E}$ and for any X, Y in $\mathbf{E}$, $[\mathbf{E}](X,Y) = \Gamma_!(Y^X)$. In the case of Ball Complexes $\Gamma_!(Y^X)$ is the set of homotopy classes of maps $X \longrightarrow Y$, where two maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are homotopic if there is a map $B_1 \xrightarrow{F} Y^X$ such that $F \cdot \delta_0 = f$ and $F \cdot \delta_1 = g$. F is called a homotopy from f to g.

Proposition: The connected components functor of $\mathcal{E}$, $\Gamma_!$, preserves finite products.

Proof: The truncation $\mathbf{B}_1$ of $\mathbf{B}$, $\boxed{B_0 \rightleftarrows B_1} \overset{i}{\hookrightarrow} \mathbf{B}$ is a full subcategory whose presheaf category is the topos of reflexive graphs. Its connected components functor π_0 (being the object part of a reflexive coequalizer) preserves finite products (see [2]). As a consequence of the lemma on page 19, we shall see that for any space $X \in \mathcal{E}$, the colimit $\Gamma_!(X)$ depends only upon its restriction $i^*(X)$ to reflexive graphs; more precisely: $\Gamma_! = \pi_0 i^*$.

Axiom 2: The subobject classifier Ω of $\mathbf{E}$ is connected: $\Gamma_!(\Omega) = 1$.

We prove $\Gamma_!(\Omega_{\mathcal{E}}) = 1$. In general if $\mathbf{E} = \mathcal{S}^{\mathbf{C}^{op}}$ and $X \in \mathbf{E}$, then $\Gamma_!(X) = 1$ if and only if the category $\mathbf{C}/X$ is connected, where $\mathbf{C}/X$ is the category of $\mathbf{C}$ figures of X.

Suppose $X = \Omega_{\mathcal{E}}$ and we have an $n-ball$ of Ω, i.e., a sieve R on B_n. Then for any point of B_n, $B_0 \xrightarrow{\delta_i} B_n$, we have $R \cdot \delta_i \in \Omega_{\mathcal{E}}(B_0)$, a point of Ω. Hence any $n-ball$ can be connected to a point and so we show that both points of $\Omega_{\mathcal{E}}$ can be connected; these points are the empty sieve and maximal sieve $\mathbf{B}(-, B_0)$ on B_0. Consider the diagram below where $< \delta_k >$ is the sieve generated by δ_k

$$\begin{tikzcd}
\emptyset \arrow[d, equal] & & {\delta_0}^*(< \delta_0 >) \arrow[d, equal] \\
{\delta_1}^*(< \delta_0 >) \arrow[r] \arrow[d] & < \delta_0 > \arrow[d, hook] & B_0 \arrow[l] \arrow[d, equal] \\
B_0 \arrow[r, hook, "\delta_1"] & B_1 & B_0 \arrow[l, hook', "\delta_0"']
\end{tikzcd}$$

The diagram shows that $\emptyset = {\delta_1}^*(< \delta_0 >)$, $B_0 = {\delta_0}^*(< \delta_0 >)$, hence, the top and bottom of Ω, $1 \underset{1}{\overset{0}{\rightrightarrows}} \Omega$ can be connected via $I \xrightarrow{<\delta_0>} \Omega$; i.e., we have

$$\begin{tikzcd}
1 \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] \arrow[rr, bend left=50, "1"] \arrow[rr, bend right=50, "0"'] & I \arrow[r, "<\delta_0>"] & \Omega
\end{tikzcd}$$

Since Ω is not only connected but also has the structure of a monoid with zero, Ω^X is connected for any X. Hence any space X can be embedded into a contractible space via $X \xhookrightarrow{\{\}_X} \Omega^X$ (the "singleton" map).

Since both the topos $\mathcal{S}^{\mathbf{S}^{op}}$, presheaves on the category $\mathbf{S}$ of finite non-empty sets (see [7] for more details about this topos), and the topos $\mathcal{S}^{\Delta^{op}}$ of simplicial sets satisfy the three axioms (the verifications are similar to those given for Ball Complexes), they are thus "toposes of spaces". Although these toposes are themselves of interest, they shall be used here primarily for comparison with $\mathcal{E}$. These two topoi share some common features. For example, both contain as reflective subcategories the cartesian closed category $\mathcal{G}$ of groupoids. The reflection for the inclusion in both cases is known as the Poincaré functor (which preserves finite products). Another well known cartesian closed subcategory of $\mathcal{S}^{\mathbf{S}^{op}}$ is the category of simplicial complexes. In fact, this is the cartesian closed category of double-negation separated objects in $\mathcal{S}^{\mathbf{S}^{op}}$. Furthermore, we shall see later that certain (similarly defined) endofunctors of both topoi behave equally well with respect to preservation of particular limits and colimits. In contrast, however, $\mathcal{S}^{\mathbf{S}^{op}}$ is codiscretely generated (while $\mathcal{S}^{\Delta^{op}}$ is not) because

$$\mathbf{S} \hookrightarrow \mathcal{S} \xhookrightarrow[\Gamma^!]{} \mathcal{S}^{\mathbf{S}^{op}},$$

the restriction of $\Gamma^!$ to finite sets, is the Yoneda embedding. Since any presheaf topos is generated by representables, we see that $\mathcal{S}^{\mathbf{S}^{op}}$ is codiscretely generated. Although this is not true of simplicial sets, in this section we shall show that in both toposes any object is a quotient of a "$1 - coskeletal$" object.

The topos $\mathcal{S}^{\mathbf{S}^{op}}$ is also known as the Boolean algebra classifier. Its role in the category of toposes defined over sets is to classify Boolean algebras in these topoi. More precisely, there is a natural equivalence of categories

$$\frac{\mathcal{F} \longrightarrow \mathcal{S}^{\mathbf{S}^{op}}}{Bool.\ alg.\ (\mathcal{F})}$$

between geometric morphisms $\mathcal{F} \longrightarrow \mathcal{S}^{\mathbf{S}^{op}}$ and Boolean algebra objects in $\mathcal{F}$, for any $\mathcal{F}$ over sets. Since $2 \in \mathbf{S}$ has the structure of a Boolean algebra (there is a switching map $2 \xrightarrow{\neg} 2$ which is not present in Δ), the presheaf $\Gamma^!(2) = 2^{(\ )}$ inherits this structure (since $\Gamma^!$ preserves products) and therefore is a Boolean algebra object of $\mathcal{S}^{\mathbf{S}^{op}}$. For any map of topoi $\mathcal{F} \xrightarrow{f} \mathcal{X}$, we take $f^*(\Gamma^!(2))$ as the Boolean algebra classified by f. The topos of simplicial sets has the property of classifying linear orders in topoi defined over sets. Here the generic linear order is given by $\Delta(-,[1])$.

It's worth pointing out here that it is not only toposes of spaces which are both local and essential over sets; there are other categories which are not topoi but have this property. For example, the category $\mathcal{G}$ of groupoids has this property:

$$\pi_0 \dashv disc. \dashv pts. \dashv codisc. \qquad \begin{tikzcd} \mathcal{G} \arrow[d, shift right=4.5] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=0, "\dashv"] \arrow[d, shift left=1.5] \arrow[d, phantom, shift left=3, "\dashv"] \\ \mathcal{S} \arrow[u, shift right=1.5] \arrow[u, shift left=4.5] \end{tikzcd}$$

The codiscrete/chaotic inclusion sends any set S to the groupoid that has exactly one map $i \longrightarrow j$ between any two objects i, j of S. Its left adjoint, pts. sends a groupoid $\mathbf{G}$ to its set of objects $\mathcal{G}\ (\mathbf{1},\ \mathbf{G})$, where $\mathbf{1}$ is the terminal, one object groupoid. The discrete functor, applied to any set S, gives the groupoid with objects the elements of S and whose maps are all identity maps. For any $\mathbf{G} \in \mathcal{G}$, $\pi_0(\mathbf{G})$ is the connected components of $\mathbf{G}$, where any two objects $g, h \in \mathbf{G}$ are in the same component iff they are isomorphic. Hence we see that the categories of discrete and codiscrete groupoids are equivalent (being equivalent to $\mathcal{S}$) but are totally opposite as subcategories in $\mathcal{G}$. Also, observe that functors

$$codisc(2) = \bullet \rightleftarrows \bullet \longrightarrow \mathbf{G}$$

pick out the arrows of $\mathbf{G}$, and functors

$$codisc(3) = \begin{tikzcd} & \bullet \arrow[dl, shift right] \arrow[dr, shift left] & \\ \bullet \arrow[ur, shift right] \arrow[rr, shift right] & & \bullet \arrow[ul, shift left] \arrow[ll, shift right] \end{tikzcd} \longrightarrow \mathbf{G}$$

tell us what arrows compose and what their composite is. Hence any groupoid $\mathbf{G}$ is a quotient of a sum of codiscrete groupoids. Another way to see this is via the inclusion $\mathcal{G} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}}$, induced by $\mathbf{S} \xhookrightarrow{chaotic} \mathcal{G}$. Any G in $\mathcal{G}$, thought of as an object of $\mathcal{S}^{\mathbf{S}^{op}}$, can be covered by codiscrete spaces. Applying the Poincaré reflection, we see $\mathbf{G}$ as quotient of chaotic groupoids.

In this section we begin the Hegelian analysis of the study of levels in our topos $\mathcal{E}$. First, we build the framework for such an analysis for a general topos and then specialize to $\mathcal{E}$.

A level in a category $\mathbf{C}$ is a functor from $\mathbf{C}$ to a smaller category $\mathbf{A}$ which has left and right adjoint sections. This is also known as a unity and identity of opposites UIO (see [5], [6] and [9]) because the two inclusions of $\mathbf{A}$ are as categories identical to $\mathbf{A}$ but united in the larger $\mathbf{C}$ are opposite as described by the adjointness. Hence the example of Groupoids with its discrete and codiscrete inclusions of sets is a UIO.

$$\begin{tikzcd}
\mathbf{C} \arrow[loop left, "sk_{\mathbf{A}}"] \arrow[loop right, "cosk_{\mathbf{A}}"] \arrow[d] \\
\mathbf{A} \arrow[u, shift left=3] \arrow[u, shift right=3] \arrow[u, phantom, shift left=1.5, "\dashv"] \arrow[u, phantom, shift right=1.5, "\dashv"]
\end{tikzcd}$$

Returning to the level $\mathbf{A}$ in $\mathbf{C}$ and composing functors, we obtain two endofunctors of $\mathbf{C}$. These endofunctors are called the coskeleton and skeleton functors for this level, or $cosk_{\mathbf{A}}$, $sk_{\mathbf{A}}$, respectively. A level $\mathbf{B}$ is lower than level $\mathbf{A}$ if, for example, the skeletal inclusion of $\mathbf{B}$ into $\mathbf{C}$ factors through the skeletal inclusion of $\mathbf{A}$ into $\mathbf{C}$. A sharper distinction can be made between these levels if we declare $\mathbf{B}$ to be qualitatively lower than $\mathbf{A}$ if its skeletal and coskeletal inclusions into $\mathbf{C}$ both factor through the coskeletal inclusion of $\mathbf{A}$ into $\mathbf{C}$. This is equivalent to $cosk_{\mathbf{A}}(sk_{\mathbf{B}}) = sk_{\mathbf{B}}$ and $cosk_{\mathbf{A}}(cosk_{\mathbf{B}}) = cosk_{\mathbf{B}}$. Let us denote this relationship between $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{B}\mathrel{\underset{\mathbf{C}}{<<}}\mathbf{A}$. If for a given $\mathbf{B}$ there is a smallest level $\mathbf{A}$ for which $\mathbf{B}\mathrel{\underset{\mathbf{C}}{<<}}\mathbf{A}$, then we say that level $\mathbf{A}$ is the Aufhebung of level $\mathbf{B}$(see [5], [6] and [9]). (By "smallest" we mean among the partially ordered class of all levels in $\mathbf{C}$).

Suppose now $\mathbf{C}$ is a topos $\mathcal{X}$ defined over a topos $\mathcal{X}_0$. Consider the poset of levels in $\mathcal{X}$ and let us assume these adjunctions are enriched in $\mathcal{X}_0$, where $\mathcal{X}_0$ is considered as the smallest nontrivial level. For any level $\mathcal{X}_n$, the right coskeletal inclusion of $\mathcal{X}_n$ is the sheaf inclusion of an essential subtopos. The left skeletal inclusion, by contrast, is usually not a subtopos, and we think of an object of this category as negating a sheaf for this level. In the previous examples, we can (via geometric realization) picture the $n - skeleton$ of a space X as that part of X consisting of things of dimensions

$\leq n$, whereas the $n-$ *coskeleton* of X has all the holes of dimension $\geq n$ in X filled in.

We now would like to assign dimensions to levels. An object X of $\mathcal{X}$ has dimension less than or equal to that of a level if the skeleton for this level fixes X. Let $\mathbf{1}$ be the terminal category with its left and right inclusions into $\mathcal{X}$ being the inclusion of the initial and terminal objects. We say that $\emptyset$ has dimension $-\infty$, and if $\emptyset$ is a sheaf for the right codiscrete inclusion of $\mathcal{X}_0$ in $\mathcal{X}$, then the Aufhebung of dimension $-\infty$ is dimension 0, with the $0-$ *dimensional* spaces being the discrete inclusion of $\mathcal{X}_0$ in $\mathcal{X}$. Furthermore, if the discrete inclusion has a left adjoint components functor π_0, then dimension 1, defined as the Aufhebung of dimension 0 (i.e., the smallest level $\mathcal{X}_n$ for which $n \underset{\mathcal{X}}{\gg} 0$), can also be described as the smallest level $\mathcal{X}_n$ for which

$$\pi_0 sk_n X = \pi_0 X \qquad \forall X \in \mathcal{X}.$$

The latter follows from the following

Proposition: Suppose that π_0 exists:

$$\begin{tikzcd}[column sep=1.5em]
\mathcal{X} \arrow[d, shift right=6, "\pi_0"'] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=0, "\dashv"] \arrow[d, phantom, shift left=3, "\dashv"] \arrow[d, shift left=1.5, leftarrow] \arrow[d, shift left=4.5] \arrow[d, shift left=7.5, leftarrow] \arrow[d, shift right=1.5, leftarrow] \\
\mathcal{X}_0
\end{tikzcd}$$

Then $n \underset{\mathcal{X}}{\gg} 0$ if and only if $\forall\, X \in \mathcal{X}$, $\pi_0 sk_n X \simeq \pi_0 X$.

Proof: $\Rightarrow$: $\forall\, X \in \mathcal{X}, Y \in \mathcal{X}_0$

$$\begin{array}{rcl}
\pi_0(sk_n(X)) & \longrightarrow & Y \\
\hline
sk_n(X) & \longrightarrow & 0^*(Y) \\
\hline
X & \longrightarrow & cosk_n(0^*(Y)) \\
\hline
X & \longrightarrow & 0^*(Y)
\end{array}$$

By uniqueness of adjoints, $\pi_0 \, sk_n = \pi_0$.

$\Leftarrow$: $\forall\, X \in \mathcal{X}, Y \in \mathcal{X}_0$

$$\begin{array}{rcl}
X & \longrightarrow & cosk_n(0^*(Y)) \\ \hline
sk_n(X) & \longrightarrow & 0^*(Y) \\ \hline
\pi_0(sk_n(X)) & \longrightarrow & Y \\ \hline
\pi_0(X) & \longrightarrow & Y
\end{array}$$

Again by uniqueness, $cosk_n 0^* = 0^*$

Usually sheaf inclusions do not preserve sums, but we see here that if a level n is qualitatively higher than dimension zero, the functor n_* preserves all sums.

Proposition: For $n \underset{\mathcal{X}}{\gg} 0$, n_* preserves all coproducts.

Proof: Observe that $n \underset{\mathcal{X}}{\gg} 0$ implies n_* preserves discretes; that is, for any discrete S, $n_*(S) = S$. Let A_i, $i \in S$, be a set of objects of $\mathcal{E}_n$. Let $A_j \overset{\mu_j}{\hookrightarrow} \sum\limits_S A_i$ be the j^{th} summand inclusion. If we apply n_* to the diagram

$$\begin{tikzcd}
A_j \arrow[r, hook, "\mu_j"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum\limits_S A_i \arrow[d] \\
1 \arrow[r, hook] & S
\end{tikzcd}$$

we get

$$\begin{tikzcd}
n_*(A_j) \arrow[r, hook, "n_*(\mu_j)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & n_*\left(\sum\limits_S A_i\right) \arrow[d] \\
1 \arrow[r, twoheadrightarrow] & S
\end{tikzcd},$$

a pullback since n_* preserves all inverse limits. Since the diagram is a pullback for any $j \in S$, $n_*\left(\sum_S A_i\right)$ is the coproduct of the $n_*(A_i)$, $i \in S$; i.e.,

$$n_*\left(\sum_S A_i\right) = \sum_S n_*(A_i).$$

Recall that any lex functor $\mathbf{C} \xrightarrow{F} \mathcal{X}$ between a small lex category $\mathbf{C}$ and a lex cocomplete category $\mathcal{X}$ induces an adjunction $R : \mathcal{X} \rightleftarrows \mathcal{S}^{\mathbf{C}^{op}} : L$ with $L \dashv R$ and L a lex functor. Here R is the functor defined $\forall\ X \in \mathcal{X},\ C \in \mathbf{C}$ as $R(X)(C) = \mathcal{X}(F(C), X)$. If we take F to be $\mathcal{X}_n \xrightarrow{n_*} \mathcal{X}$, then the result is a geometric morphism $\mathcal{X} \rightleftarrows \mathcal{X}_0^{\mathcal{X}_n^{op}}$. If the direct-image functor is full and faithful then we say that $\mathcal{X}$ is $n-generated$, for then every object of $\mathcal{X}$ is a colimit of n_* values, i.e., a quotient of a sum of $n-coskeletal$ objects. In this way we may think of $\mathcal{X}$ as analogous to the sheaves for a subcanonical topology on $\mathcal{X}_n$ even if $\mathcal{X}$ itself is not a presheaf topos.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift left] & \mathcal{X}_0^{\mathcal{X}_n^{op}} \arrow[l, shift left] \\
\mathcal{X}_n \arrow[u, "n_*"] \arrow[ur] &
\end{tikzcd}$$

Let us now return to the topos $\mathcal{E}$ of Ball Complexes. We would like to study the levels (essential subtoposes) in $\mathcal{E}$. However, we need to know exactly what all the essential subtoposes are before we can compare these levels. From [8] we see that the essential subtoposes of $\mathcal{E}$ are all of the form $\mathcal{S}^{\mathbf{C}^{op}}$ where $\mathbf{C} \hookrightarrow \mathbf{B}$ is a full subcategory of $\mathbf{B}$ closed under the splitting of idempotents (the same is true of $\mathcal{S}^{\mathbf{S}^{op}}$ and $\mathcal{S}^{\Delta^{op}}$). The full subcategories of $\mathbf{B}$ are simply the truncations of $\mathbf{B}$ at $n \in \mathbf{N}$ for some n (this is also true of both $\mathbf{S}$ and Δ). Let $\mathcal{E}_n = \mathcal{S}^{\mathbf{B}_n{}^{op}}$ and follow the notation in the diagram:

$$n_! \dashv n^* \dashv n_* \qquad \begin{tikzcd} \mathcal{S}^{\mathbf{B}^{op}} \arrow[d, shift left=2ex, "n^*"] \\ \mathcal{S}^{\mathbf{B}_n^{op}} \arrow[u, shift left=0ex, "n_!"] \arrow[u, shift right=4ex, "n_*"'] \end{tikzcd}$$

Lemma: For $n > 0$, $B_r \in \mathbf{B}$, $n^*(B_r)$ is connected.

Proof: Any $m - ball$ of $n^*(B_r)$ can be connected to a point via any $1 \xhookrightarrow{\delta_k} B_r$, so we show that both points can be joined. Since, for example, $\delta_1 \cdot \delta_i = \delta_i$, we see that δ_1 connects the two points (see diagram)

$$\begin{tikzcd} 1 \arrow[r, hook, shift left, "\delta_0"] \arrow[r, hook, shift right, "\delta_1"'] \arrow[rr, bend left=40, "0"] \arrow[rr, bend right=40, "1"'] & B_1 \arrow[r, hook, "\delta_1"] & B_m. \end{tikzcd}$$

We shall now show that the two definitions of level 1 are equivalent.

Proposition: If $n > 0$, then n_* preserves coproducts.

Proof: Let X_j, $j \in J$ be a set of objects of $\mathcal{E}_n$. For any $B_s \in \mathbf{B}$ we have

$$\begin{array}{cl}
1 \longrightarrow n_*\left(\sum_J X_j\right)(B_s) & \\ \hline
B_s \longrightarrow n_*\left(\sum_J X_j\right) & Yoneda \\ \hline
n^*(B_s) \longrightarrow \sum_J X_j & n^* \dashv n_* \\ \hline
n^*(B_s) \longrightarrow X_i & n^*(B_s) \text{ is connected} \\ \hline
B_s \longrightarrow n_*(X_i) & n^* \dashv n_* \\ \hline
1 \longrightarrow n_*(X_i)(B_s) & Yoneda \\ \hline
1 \longrightarrow \sum_J n_*(X_j)(B_s) & 1 \text{ is connected}
\end{array}$$

Hence $n_*\left(\sum_j X_j\right)(B_s) = \sum_j n_*(X_j)(B_s)$ and we conclude that $n_*\left(\sum_J X_j\right) = \sum_J n_*(X_j)$. Since the collection X_j, $j \in J$ was arbitrary, the result follows.

It follows (from adjointness) that $\Gamma_!(X) = \pi_0(n^*(X))$ for $X \in \mathcal{E}$, $n > 0$. In particular, we may take $n = 1$ and hence the set of components of a ball complex is the set of components of the associated reflexive graph. (See below for a brief discussion on reflexive graphs).

The lemma and hence the proposition are both true for $\mathcal{S}^{\mathbf{S}^{op}}$ and $\mathcal{S}^{\Delta^{op}}$ and we can say more for these toposes. First, let us observe another feature that simplicial sets and Ball Complexes have in common. If we truncate Δ and $\mathbf{B}$ at the level of the second object, we get a category $1 \;\substack{\xrightarrow{\delta_0} \\ \xleftarrow{p} \\ \xrightarrow[\delta_1]{}}\; I$ where I denotes the interval. The topos of presheaves on this category is called reflexive graphs. A typical graph X has edges $I \xrightarrow{a} X$, with the source and target vertices of a given by the two composites

$$\begin{tikzcd} 1 \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] \arrow[dr, shift left] \arrow[dr, shift right] & I \arrow[d, "a"] \\ & X \end{tikzcd}$$

The arrows $I \longrightarrow 1 \longrightarrow X$ are the degenerate edges. In particular we can picture reflexive graphs with every edge an arrow connecting two degenerate edges denoted by solid dots. I can be pictured as $I = \bullet \longrightarrow \bullet$. If we truncate $\mathbf{S}$ at 2 we get the category $\mathbf{S}_1 : 1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; 2 \xrightarrow{\tau} 2$, where τ is the interchange map. An object X of $\mathcal{S}^{\mathbf{S}_1^{op}}$ can also be pictured as having edges and dots, but here the presence of τ implies that for any edge a there is a canonical companion edge $a \cdot \tau$ that travels in the opposite direction; hence this topos is called two-way reflexive graphs. The next proposition shows that not only is the topos of simplicial sets $1-$*generated*, but every object in it can be covered by a $1-$*coskeletal* object.

Proposition (Lawvere): The topos of simplicial sets is $1-$*generated*.

First a little discussion and notation. For any of the three toposes under consideration the $n-skeleton$ of a space X at stage m, $sk_n(X)_m$, consists of all figures of X of shape m ($m-simplices$, $m-balls$, etc.) that are degenerated from a figure of X of shape q where $q \leq n$. In particular, $sk_n(p)$ is that subobject of p consisting of maps that factor through some $n \hookrightarrow p$. Let Δ_n denote the truncation of Δ at n.

Proof: The first thing we prove is that for $n \geq 1$, $cosk_n(m) = m \quad \forall\, m \in \Delta$.

Let $p \in \Delta$ with $p > n$. By adjointness natural maps $p \longrightarrow cosk_n(m)$ correspond to maps $sk_n(p) \longrightarrow m$. We must show that any map $sk_n(p) \xrightarrow{f} m$ extends uniquely to a $p \xrightarrow{\overline{f}} m$ as in the diagram:

$$\begin{tikzcd} p \arrow[dr, "\exists!\overline{f}"] & \\ sk_n(p) \arrow[u, hook] \arrow[r, "\forall f"] & m \end{tikzcd}$$

Taking $\overline{f}$ to be the component of f at 0, $p \xrightarrow{f_0} m$, defines the unique extension. Hence maps $p \longrightarrow cosk_n(m)$ are precisely maps $p \longrightarrow m$, and by Yoneda $cosk_n(m) = m$. Let $X \in \mathcal{S}^{\Delta^{op}}$. Since the representables generate any presheaf topos, there are objects $n_j \in \Delta$ for j in some set J and an epimorphism $\sum\limits_j n_j \longrightarrow X$. But for $n > 0$, $\sum\limits_j n_j = \sum\limits_j n_* n^*(n_j) = n_*\left(\sum\limits_j n^*(n_j)\right)$, the last equality being true since n_* preserves sums. Hence $n_*\left(\sum\limits_j n^*(n_j)\right) \twoheadrightarrow X$, and so simplicial sets are $n-generated$.

In particular, simplicial sets are $1-generated$ and any simplicial set can be covered by a $1-coskeletal$ simplicial set coming from a reflexive graph. We know that $\mathcal{S}^{\mathbf{S}^{op}}$ is codiscretely generated, but it is also true that any $X \in \mathcal{S}^{\mathbf{S}^{op}}$ is the quotient of a $1-coskeletal$ object. In fact, for any $n \geq 0$ and $S \in \mathbf{S}$, $cosk_n(S) = S$. The codiscrete inclusions in either topos, however, do not preserve sums.

Lawvere has pointed out the following construction. Suppose that $\mathcal{X}$ is a topos in which a level $\mathcal{X}_\alpha$ generates and α_* preserves coproducts. Let $X \in \mathcal{X}$, cover X with an $\alpha - coskeletal$ object and take the kernel pair

$$Y \underset{\pi_1}{\overset{\pi_0}{\underset{\leftarrow \delta \supset}{\rightrightarrows}}} \alpha_*(W) \twoheadrightarrow X,$$

where $\pi_i \circ \delta = 1$. We can also cover Y with an $\alpha - coskeletal$ object $\alpha_*(D) \xrightarrow{\gamma} Y$ and thus obtain the following diagram

$$\alpha_*(D+W) = \alpha_*(D) + \alpha_*(W) \underset{\beta_1}{\overset{\beta_0}{\underset{\leftarrow \mu \supset}{\rightrightarrows}}} \alpha_*(W) \twoheadrightarrow X,$$

where $\beta_i = \pi_i \circ (\gamma + \delta)$. Since $\gamma + \delta$ is an epimorphism, and since the first diagram is a coequalizer diagram, the second diagram is also a coequalizer diagram. Moreover, $(\gamma + \delta) \circ \mu = \delta$ and hence $\beta_i \circ \mu = \pi_i \circ \delta = 1$. Since α_* is a full inclusion, β_i and μ come from arrows in $\mathcal{X}_\alpha$. Thus we have a functor

$$\mathcal{X}_\alpha{}^{\Delta_1{}^{op}} \xrightarrow{coeq\ \circ \alpha_*} \mathcal{X}$$

which preserves finite products and all coproducts. When $\mathcal{X} = \mathcal{S}^{\Delta^{op}}$, we can take $\mathcal{X}_\alpha$ to be the topos of reflexive graphs. Thus simplicial sets receive a surjective functor from a graphics topos

$$\mathcal{S}^{(\Delta_1 \times \Delta_1)^{op}} = (\mathcal{S}^{\Delta_1{}^{op}})^{\Delta_1{}^{op}} \twoheadrightarrow \mathcal{S}^{\Delta^{op}}.$$

In sharp contrast to the $0 - generation$ of $\mathcal{S}^{\mathbf{S}^{op}}$ and $1 - generation$ of $\mathcal{S}^{\Delta^{op}}$, $\mathcal{S}^{\mathbf{B}^{op}}$ is not $n - generated$ for any n. Before we can prove this we need some general remarks about presheaf topoi.

Lemma: In any presheaf category, the representables are projective and connected.

Lemma: Let $\mathbf{C}$ be a small category and J a Grothendieck topology on $\mathbf{C}$. If $X \in \mathcal{S}^{\mathbf{C}^{op}}$ is a sheaf for this topology, then any retract of X is also a sheaf.

Proposition: Let $\mathcal{X}$ be a presheaf topos $\mathcal{S}^{\mathbf{C}^{op}}$ and suppose that all of its essential subtoposes $\mathcal{X}_n$ are of the form $\mathcal{S}^{\mathbf{C}_n{}^{op}}$, where $\mathbf{C}_n \hookrightarrow \mathbf{C}$ is a full subcategory of $\mathbf{C}$. The following are equivalent:

$(a) \quad \mathcal{X}\ is\ n-generated$

$(b) \quad cosk_n(C) = C \qquad \forall\, C \in \mathbf{C}$

Proof: (**a**) $\Rightarrow$ (**b**). Let $C \in \mathbf{C}$. Since $\mathcal{X}$ is $n - generated$, there is a collection of objects $A_\alpha \in \mathcal{X}_n$, $\alpha \in I$ for some set I and epimorphism

$$\sum_{\alpha \in I} n_*(A_\alpha) \xrightarrow{\theta} C.$$

Since C is projective, there is a section ρ for θ.

But the domain of ρ is the connected C and therefore ρ must factor through a summand $n_*(A_\beta)$ for some $\beta \in I$:

$$\begin{tikzcd}
C \arrow[r, hook, shift left, "\rho"] \arrow[dr, shift left] & \sum_{\alpha \in I} n_*(A_\alpha) \arrow[l, two heads, shift left, "\theta"] \\
& n_*(A_\beta) \arrow[u, hook] \arrow[ul, shift left]
\end{tikzcd}$$

Therefore, C is a retract of $n_*(A_\beta)$ and is therefore $n-coskeletal$, i.e., $cosk_n(C) = C$.

(**b**) $\Rightarrow$ (**a**). If $X \in \mathcal{S}^{\mathbf{C}^{op}}$, then there are representables $C_\alpha \in \mathbf{C}$, $\alpha \in I$ for some set I and an epimorphism $\sum_\alpha C_\alpha \xrightarrow{\theta} X$. Since $n_* n^*(C_\alpha) = C_\alpha$, we have $\sum_\alpha n_*(n^*(C_\alpha)) \longrightarrow X$ and hence X is covered by a sum of values of n_*.

We now show that $\mathcal{E} = \mathcal{S}^{\mathbf{B}^{op}}$ is not $n - generated$ by showing, equivalently, there is no n for which every representable is a sheaf for the level n.

Lemma: Let $\mathcal{E}$ be the topos of Ball Complexes. If $n \geq 0$, and $m \geq n+2$, then $cosk_n(B_m) \neq B_m$.

Proof: For B_m to be $n - coskeletal$ for any $B_s \in \mathbf{B}$, we must uniquely extend the arrow φ in the diagram

$$\begin{tikzcd} B_s \arrow[dr, "\overline{\varphi}"] & \\ sk_n(B_s) \arrow[u, hook] \arrow[r, "\varphi"] & B_m \end{tikzcd}$$

to a $\overline{\varphi}$. Choose $m > s > n$. Any φ is determined by its value at the two inclusions $B_n \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} B_s$ (since they generate $sk_n(B_s)$). Define φ by $\varphi(\delta_0) = \delta_0$, $\varphi(\delta_1) = \delta_1$. Then there are two extensions $\overline{\varphi} = \delta_0$ and $\overline{\varphi} = \delta_1$ for φ.

However, we do have the following

Lemma: For $n > 0$, $cosk_n(B_n) = B_n$

Proof: Let $k > n$ and consider a map $sk_n(B_k) \xrightarrow{\varphi} B_n$. Let $B_n \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} B_k$ be the hemispherical inclusions which "generate" $sk_n(B_k)$. Let

$$\varphi_n(\delta_i) = f_i : B_n \longrightarrow B_n.$$

By naturality $f_0 \cdot \delta = f_1 \cdot \delta$ for every $B_s \overset{\delta}{\hookrightarrow} B_n$, and therefore $f_0 = f_1$; we may take $\overline{\varphi} = f_0 \cdot p$ and this is obviously unique.

In particular, $cosk_1(B_1) = B_1$ and, therefore, $cosk_n(B_1) = B_1 \quad \forall n > 0$. This implies that the $n - coskeletal$ endofunctors preserve homotopies. In a moment, we shall see that sk_n has a left adjoint. Hence, since $sk_n(B_1) = B_1$ for $n > 0$, and since sk_n therefore preserves binary products for $n \in \mathbf{N}$, we see that the $n - skeletal$ endofunctors preserve homotopies for $n > 0$.

For any presheaf topos $\mathcal{X}$ the topos of sheaves for the canonical topology is defined to be the smallest subtopos of $\mathcal{X}$ for which the representables are sheaves. Hence we see that the topos of canonical sheaves for $\mathcal{S}^{\mathbf{S}^{op}}$ is $\mathcal{S}$, where the sheaf inclusion is the codiscrete functor $\Gamma^!$. In the case of simplicial sets all representables are $1 - coskeletal$, therefore, the canonical sheaves are $1 - coskeletal$ (reflexive graphs). Since the representables in $\mathcal{S}^{\mathbf{B}^{op}}$ are not $n - coskeletal$ for any given n, we see that the canonical topology is the trivial one.

If $\mathcal{X}$ is either $\mathcal{S}^{\mathbf{S}^{op}}$ or $\mathcal{S}^{\Delta^{op}}$, we have seen that the $n - coskeleton$ functors preserve all coproducts if $n > 0$. The $n - skeletal$ functors have good properties also, for they preserve equalizers for any n. The situation is even stronger for Ball Complexes since here the skeletal inclusion $n_!$ has a left adjoint π_n for all n : this functor is induced by a functor p_n, left adjoint to the inclusion

$$\begin{tikzcd} \mathbf{B} \arrow[d, shift right=2, "p_n"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{B}_n \arrow[u, hook, shift right=2, "i_n"'] \end{tikzcd} \quad ,$$

where p_n is defined on objects as $p_n(B_k) = B_n$ for $k > n$ and $p_n(B_k) = B_k$ if $k \leq n$. That this defines a functor follows from showing $p_n \dashv i_n$; we show that $B_s \xrightarrow{\eta_s} i_n p_n(B_s)$ is the unit of the adjunction where η_s is the identity if $s \leq n$ and the unique map if $s > n$. If $s > n$ and $B_s \longrightarrow i_n(B_k)$, then we may factor this uniquely as $B_s \twoheadrightarrow B_n \longrightarrow i_n(B_k)$, with the first map the unit. The case with $s \leq n$ is clear.

Taking the left Kan extension π_n of $\mathbf{B} \xrightarrow{p_n} \mathbf{B}_n$, we have $\pi_n = p_{n_!}$, $n_! = p_n^*$, and $n^* = p_{n_*}$ which we may picture as

$$\begin{array}{cc}
 & \begin{tikzcd} \mathcal{S}^{\mathbf{B}^{op}} \arrow[d, shift right=6] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, shift left=0, "\dashv"] \arrow[d, shift left=2] \arrow[d, phantom, shift left=4, "\dashv"] \\ \mathcal{S}^{\mathbf{B}_n^{op}} \arrow[u, hook, shift right=2] \arrow[u, hook, shift left=6] \end{tikzcd} \\
\pi_n \dashv n_! \dashv n^* \dashv n_* & \\
 & \begin{tikzcd} \mathcal{S}^{\mathbf{B}_n^{op}} \arrow[d, shift right=6] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, shift left=0, "\dashv"] \arrow[d, shift left=2] \arrow[d, phantom, shift left=4, "\dashv"] \\ \mathcal{S} \arrow[u, hook, shift right=2] \arrow[u, hook, shift left=6] \end{tikzcd} \\
\Gamma_! \dashv \Gamma^* \dashv \Gamma_* \dashv \Gamma^! &
\end{array}$$

We thus have $n_!\pi_n \dashv sk_n \dashv cosk_n$, and hence sk_n preserves all inverse limits as well as all direct limits. Of course the same is true of $n_!$, and later in this section we shall use its preservation of finite products.

Let us return to the analysis of the levels in the topos of Ball Complexes. We begin by calculating the Aufhebung of level n. First, we shall need a lemma.

Lemma: If $X \in \mathcal{S}^{\mathbf{B}^{op}}$ and x and y are degenerate $n - balls$ of X that have a hemisphere in common, then x and y are equal.

Proof: We may write $x = x' \cdot p$ and $y = y' \cdot p$ for some $(n-1) - balls$ x', y'. Since $x \cdot \delta_i = y \cdot \delta_i$ for some $B_{n-1} \overset{\delta_i}{\hookrightarrow} B_n$, we have $x' = x \cdot \delta_i = y \cdot \delta_i = y'$. Hence $x = y$.

Proposition: The Aufhebung function is the successor function.

Proof: Let $X \in \mathcal{S}^{\mathbf{B}^{op}}$ and suppose that X is $n - skeletal$. We shall prove that X is $(n+1) - coskeletal$. To be a sheaf for level $n+1$ means precisely that, for any $B_m \in \mathbf{B}$ and any $sk_{n+1}(B_m) \xrightarrow{\varphi} X$ in $\mathcal{E}$, we can uniquely extend φ to an $m - ball$

$\overline{\varphi}$ of X:

$$\begin{tikzcd} B_m \arrow[dr, "\exists ! \overline{\varphi}"] & \\ sk_{n+1}(B_m) \arrow[u, hook] \arrow[r, "\forall \varphi"] & X \end{tikzcd}$$

If $m \leq n+1$, then $sk_{n+1}(B_m) = B_m$ and the result follows trivially, so assume $m > n+1$. Observe that the sieve $sk_{n+1}(B_m)$ is generated by the two inclusions $B_{n+1} \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} B_m$; hence φ is determined by $\varphi(\delta_i)$, $i \in \{0,1\}$. Furthermore, $\varphi(\delta_i)$ are both $(n+1)-balls$ of X and are therefore degenerate since X is $n-skeletal$. Using the naturality of φ we have $\varphi(\delta_i) \cdot \delta_j = \varphi(\delta_j)$, hence $\varphi(\delta_0)$ and $\varphi(\delta_1)$ have the same upper and lower hemisphere and are therefore equal by the lemma. We may take $\overline{\varphi}$ to be $\varphi(\delta_1) \cdot p$ and again this is unique by the lemma.

Moreover, if X is an $n-skeletal$ object, then it is a quotient of a sum of representables B_i, $i \leq n$. From the lemma on page 27, we see that X is, therefore, a quotient of an $n-coskeletal$ object.

Michael Zaks, in [16], has calculated the Aufhebung function for the lattice of essential subtoposes of $\mathcal{S}^{\Delta^{op}}$. He found that Aufhebung of level n is $n+1$ for $n \leq 2$, and $2n-1$ for $n > 2$. The calculation for $\mathcal{S}^{\mathbf{S}^{op}}$ has still to be carried out.

Axiom 2 for a topos of spaces requires that Ω be connected. In the case of $\mathcal{S}^{\Delta^{op}}$ and $\mathcal{S}^{\mathbf{B}^{op}}$, Ω has the further property of being a cogenerator. Richard Squire has given neccesary and sufficient conditions on $\mathbf{C}$ for Ω in $\mathcal{S}^{\mathbf{C}^{op}}$ to cogenerate and has explicitly proved Ω cogenerates simplicial sets (see [15]). Presheaf toposes $\mathcal{S}^{\mathbf{C}^{op}}$ for which the monoid of endomaps of any C in $\mathbf{C}$ is a graphic monoid are called graphic toposes. Gustavo Arenas proved that the subobject classifier in a graphic topos cogenerates (see [10]). In particular, $\Omega \in \mathcal{S}^{\mathbf{B}^{op}}$ cogenerates.

Proposition (Gustavo Arenas): If $\mathcal{S}^{\mathbf{C}^{op}}$ is a graphic topos, then Ω co~~/~~generates.

Proof: If $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are different, then the composites $C \xrightarrow{x} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are different for some x and C. Let $z = fx, w = gx$. Let $Y \xrightarrow{\theta} \Omega$ be the characteristic map of z or w.

Then if in both cases θ does not distinguish f and g, we have $w \in < z >$, $z \in < w >$. Therefore, there are endomaps α, β of C and $w = z \cdot \alpha$, $z = w \cdot \beta$. We have

$$w = z\alpha = w\beta\alpha = w\beta\alpha\beta = z\alpha\beta = w\beta = z$$

In contrast to the Aufhebung definition we may ask the following question: for a given level n and $n-coskeletal$ inclusion into the big category, does there exist a smallest level m for which the $m-skeletal$ inclusion into the big category includes the "lower" $n-coskeletal$ sheaf inclusion? Remarkably, for $\mathcal{E}$ this m exists and is actually $n+1$. Before proving this, let us first look at these $n-coskeletal$ spaces in some more detail. We have already seen that codiscrete objects are "constant" at stages beyond B_1. It is natural to ask if something similar is true for $n-coskeletal$ objects, $n > 0$. Since we can describe the $1-coskeletal$ objects using the reflexive graphs, we give the following description: For any reflexive graph X, $1_*(X)$ can be described in terms of mapping into X; explicitly, for $B_m \in \mathbf{B}$ and $n = 1$, $1_*(X)(B_m) = \textbf{Ref. Gphs.}\ (1^*(B_m),\ X)$. We now need to see exactly what the graphs $1^*(B_m)$ look like. For $m = 0$, $1^*(B_0)$ is the graph whose vertices $1^*(B_0)(B_0) = \mathbf{B}(B_0,\ B_0)$ are one in number and has edges $1^*(B_0)(B_1) = \mathbf{B}(B_1,\ B_0)$; therefore, we may picture this graph as a dot with only a degenerate edge, $\bullet$. The graph $1^*(B_1)$ is a bit more interesting. Its vertices are $\mathbf{B}(B_0,\ B_1)$ and its edges are the elements of $\mathbf{B}(B_1,\ B_1)$. This consists of an edge $B_1 \xrightarrow{1} B_1$ with source and target the vertices $1 \cdot \delta_0 = \delta_0$ and $1 \cdot \delta_1 = \delta_1$, respectively; therefore we picture this as an arrow, $\bullet \longrightarrow \bullet$. The other two edges of $1^*(B_1)$ are the elements $B_1 \twoheadrightarrow B_0 \xhookrightarrow{\delta_i} B_1$, which begin and end at δ_i. So these are the degenerate loops at the vertices δ_i and the graph $1^*(B_1)$ is $\bullet \longrightarrow \bullet$. The graphs $1^*(B_m)$, $m > 1$, are all equal to $\bullet \rightrightarrows \bullet$ and the maps $B_m \underset{\delta_1}{\overset{\delta_0}{\leftarrow p \rightrightarrows}} B_{m+1}$ all induce isomorphisms between these graphs. Therefore, the description of $1_*(X)$ is:

$1_*(X)(B_0) =$ **Ref. Gphs.** $(\bullet, X)$, the vertices of X;

$1_*(X)(B_1) =$ **Ref. Gphs.** $(\bullet \longrightarrow \bullet, X)$, the edges of X; and

$1_*(X)(B_m) =$ **Ref. Gphs.** $(\bullet \rightrightarrows \bullet, X)$, the parallel edges of X, $m > 1$,

with the maps $B_m \underset{\delta_1}{\overset{\delta_0}{\leftarrow p \rightrightarrows}} B_{m+1}$ inducing isomorphisms between $1_*(X)(B_m)$ and $1_*(X)(B_{m+1})$. All told, $0 -$ *coskeletal* objects are $1 -$ *skeletal* and $1 -$ *coskeletal* objects are $2 -$ *skeletal*. In general we have the following

Proposition: In the topos $\mathcal{E} = \mathcal{S}^{\mathbf{B}^{op}}$, $n -$ *coskeletal* implies $(n+1) -$ *skeletal*.

Proof: We show that, for $X \in \mathcal{E}$ and $m > n$, $n_*(X)(B_m)$ and $n_*(X)(B_{m+1})$ are isomorphic. By definition, $n_*(X)(B_m) = \mathcal{E}_n(n^*(B_m), X)$ and we have

$$n^*(B_m) \underset{\delta_1{}^*}{\overset{\delta_0{}^*}{\leftarrow p^* \rightrightarrows}} n^*(B_{m+1}).$$

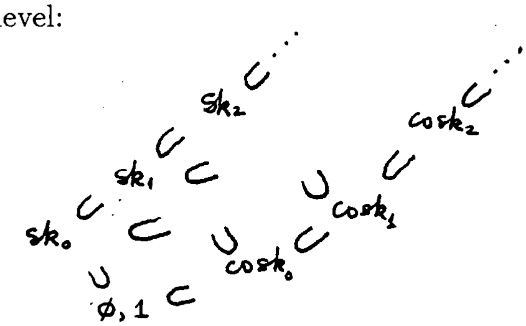
It suffices to show that $n^*(B_{m+1})$ and $n^*(B_m)$ are isomorphic. For any $B_k \xrightarrow{f} B_{m+1} \in n^*(B_{m+1})(B_k)$ and $k \leq n$, we have $\delta_i{}^*(p^*(f)) = \delta_i pf$. Since f must factor through some $B_l \xhookrightarrow{\delta_j} B_{m+1}$, $l \leq k$, we see that

$$B_k \xrightarrow{f} B_{m+1} \xrightarrow{p} B_m \xhookrightarrow{\delta_i} B_{m+1} = B_k \xrightarrow{f} B_{m+1}.$$

Therefore, $n^*(B_{m+1}) \cong n^*(B_m)$ and it follows that $(n_*X)(B_s)$ are all isomorphic for $s > n$. This means that n_*X is $(n+1) -$ *skeletal*.

Since the category of $n -$ *skeletal* objects are $(n+1) -$ *coskeletal* as well as $(n +$

1) $-$ *skeletal*, and similarly, since the category of $n-$ *coskeletal* objects are not only $(n+1)-$ *coskeletal*, but also $(n+1)-$ *skeletal*, we have the union of $n-$ *skeletons* and $n-$ *coskeletons* contained in the intersection of the $(n+1)-$ *skeletons* and the $(n+1)-$ *coskeletons*. The following is a picture of these categories and their inclusions into the next level:



For any $n \geq 0$, we also have the following picture where the inclusions have names F, G, K, L, M, N for simplicity:

$$
\begin{tikzcd}[row sep=huge]
\mathcal{E} \arrow[d] \\
\mathcal{E}_{n+2} \arrow[u, shift left=8, "F"] \arrow[u, shift right=8, "G"'] \arrow[u, phantom, shift left=4, "\dashv"] \arrow[u, phantom, shift right=4, "\dashv"] \arrow[d] \\
\mathcal{E}_{n+1} \arrow[u, shift left=8, "K"] \arrow[u, shift right=8, "L"'] \arrow[u, phantom, shift left=4, "\dashv"] \arrow[u, phantom, shift right=4, "\dashv"] \arrow[d] \\
\mathcal{E}_{n} \arrow[u, shift left=8, "M"] \arrow[u, shift right=8, "N"'] \arrow[u, phantom, shift left=4, "\dashv"] \arrow[u, phantom, shift right=4, "\dashv"]
\end{tikzcd}
$$

We have $F(KM) \cong (GL)M$ (by the Aufhebung relation)$\cong (FL)M$ (by the coAufhebung relation) $\cong F(LM)$ and therefore $KM \cong LM$; similarly, $KN \cong LN$. Thus we see that the left and right inclusions among levels have the same properties under composition as δ_0, δ_1 within $\mathbf{B}$ (even though this is not some trivial functorality).

Consider the full subcategory $T_n \hookrightarrow \mathcal{E}_n$ where the $n-$*skeletal* and $n-$*coskeletal* inclusions agree, i.e., those $X \in \mathcal{E}_n$ for which the canonical map $n_!(X) \longrightarrow n_*(X)$ is an isomorphism. Recall that for $X \in \mathcal{E}_n$

$$n_*(X)(B_{n+1}) = \mathcal{E}_n(n^*(B_{n+1}), X).$$

This means that $X \in T_n$ precisely when

$$\begin{tikzcd} n^*(B_{n+1}) \arrow[rr, "\forall f"] \arrow[dr] & & X \\ & B_n \arrow[ur, "\exists \overline{f}"'] & \end{tikzcd}$$

For $n > 0$, to give the map f is equally to give a pair of $n-$*balls* x, y of X whose hemispheres are equal: $x \cdot \delta_i = y \cdot \delta_i \quad i \in \{0,1\}$. Hence $X \in T_n$ if and only if any pair $B_n \overset{x}{\underset{y}{\rightrightarrows}} X$ with equal hemispheres are themselves equal. Let $T_n \overset{i_n}{\hookrightarrow} \mathcal{E}_n$ denote the inclusion. Given $X \in \mathcal{E}_n$, $n > 0$, "$x \sim y$ if and only if $x \cdot \delta_i = y \cdot \delta_i \quad i \in \{0,1\}$" defines an equivalence relation on $n-$*balls* of X. Hence there is a reflection $\mathcal{E}_n \xrightarrow{r_n} T_n$ with $r_n(Y)(B_n) = Y(B_n)/\sim$ for $Y \in \mathcal{E}_n$, and the unit of the adjunction $r_n \dashv i_n$ $Y \twoheadrightarrow r_n(Y)$ is an epimorphism. If Y is a product $\Pi_{i \in I} Y_i$ for $Y_i \in \mathcal{E}_n$, I a set, then $(\Pi_{i \in I} Y_i)(B_n)/\sim\ = \Pi_{i \in I}(Y_i(B_n)/\sim)$, hence r_n preserves all products. Furthermore, recall that for any reflective subcategory B of a cartesian closed category A, the reflection preserves finite products if and only if B is closed under exponentiation by all objects of A. Thus, in particular, for $n \geq 0$, T_n is closed under exponentiation by objects of $\mathcal{E}_n$, under inverse limits, and for $n > 0$, all coproducts.

When $n = 0$, $S \in T_0$ precisely when $S \overset{\Delta}{\hookrightarrow} S^2$ is an isomorphism; therefore, $T_0 = \emptyset \longrightarrow 1$. For $n = 1$, $\mathcal{E}_1$ is the topos of reflexive graphs and T_1 is the reflective subcategory $\mathcal{Y}_{\mathbf{B}_1}$ of $\mathcal{E}_1$ whose objects "live on their points": for any $X \in T$, and any pair of edges $I \overset{w}{\underset{z}{\rightrightarrows}} X$ with $w \neq z$ there is a node $1 \xrightarrow{a} I$ distinguishing them: $wa \neq za$. Here the subobject classifier Ω for reflexive graphs is $\bullet \rightleftarrows \overset{\circlearrowleft}{\bullet}$ and hence $r_1(\Omega) = \bullet \rightleftarrows \bullet$.

Of course if $\mathbf{C}$ is any category with a terminal object 1, then we may define (see [3]) $\mathcal{Y}_{\mathbf{C}}$ to be the reflective subcategory of $\mathcal{S}^{\mathbf{C}^{op}}$ whose objects X satisfy the above condition with I now an arbitrary object in $\mathbf{C}$, in particular, $\mathcal{Y}_{\mathbf{B}_n}$ makes sense for $n \in \mathbf{N}$. But for $n \neq 1$, $T_n \neq \mathcal{Y}_{\mathbf{B}_n}$. For example, the reflection $\mathcal{S}^{\mathbf{B}_n{}^{op}} \longrightarrow \mathcal{Y}_{\mathbf{B}_n}$ identifies the two $(n-1)-balls$ $B_{n-1} \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} B_n$ in B_n (among other things) since they have the same points; i.e., $B_n \notin \mathcal{Y}_{B_n}$. On the other hand, since $B_n = n_!(B_n) = n_*(B_n)$ (Lemma on p. 27), we have $B_n \in T_n$.

In any presheaf topos $\mathcal{X} = \mathcal{S}^{\mathbf{C}^{op}}$, each representable C has a maximal proper subobject $M \hookrightarrow C$. For any $D \in \mathbf{C}$, the $D-figures$ of M consists of maps $D \longrightarrow C$ that are not split epi. If $\mathcal{X}$ is either $\mathcal{S}^{\mathbf{S}^{op}}$ or $\mathcal{S}^{\Delta^{op}}$ and n is a representable, then $M = sk_{n-1}(n)$. This is also known as the boundary of n, $\partial(n)$, since the geometric realization of the pushout

$$\begin{tikzcd} \partial(n) \arrow[r, hook] \arrow[d] & n \arrow[d] \\ 1 \arrow[r, hook] & S^n \end{tikzcd}$$

is the $n-sphere$. Here the geometric realization assigns to any n the free convex space on the underlying $n+1-element$ set. In [1], it is shown that Cat is a reflective subcategory of simplicial sets

$$Cat \overset{G}{\underset{D}{\rightleftarrows}} \mathcal{S}^{\Delta^{op}}.$$

The authors also explicitly calculate $G(\partial(n))$ for all $n > 0$. It is therefore of interest to know for which levels $\partial(n)$ is a sheaf. This is answered in the following proposition.

Proposition: Let $\mathcal{X} = \mathcal{S}^{\mathbf{C}^{op}}$ be simplicial sets or the Boolean algebra classifier. For $n > 0$,

$$cosk_m(\partial(n)) = \begin{cases} n, & m < n \\ \partial(n), & m \geq n \end{cases}$$

Proof: Let $m < n$, $p \in \mathbf{C}$, then

$$\begin{array}{c} 1 \longrightarrow cosk_m(\partial(n)(p)) \\ \hline sk_m(p) \longrightarrow \partial(n) \\ \hline sk_m(p) \longrightarrow n \\ \hline p \longrightarrow n \end{array}$$

Hence, by Yoneda $cosk_m(\partial(n)) = n$.

Let $m \geq n$. We wish to show that any map φ extends uniquely to a $\overline{\varphi}$ as in the commutative diagram

$$\begin{tikzcd} sk_m(p) \arrow[r, "\varphi"] \arrow[d, hook] & \partial(n) \arrow[d, hook] \\ p \arrow[r, "\varphi_0"'] \arrow[ur, "\overline{\varphi}"] & n \end{tikzcd}$$

where the map $\overline{\varphi}$ is determined by $\varphi_0 : \ p \longrightarrow n$ (where φ_0 is the component of φ at 0) with $\varphi(f) = \varphi_0 \circ f$ (In $\mathcal{S}^{\Delta^{op}}$, this φ_0 is order-preserving). The point is to show that φ_0 belongs to $\partial(n)(p)$. If $m \geq p$ everything is trivial, so assume that $m < p$. If φ_0 is surjective, then let σ be a section of φ_0. Since the image of $n \xrightarrow{\sigma} p$ is $n \leq m$, σ belongs to $sk_m(p)$. But then $\varphi(\sigma) = \varphi_0 \cdot \sigma = 1 \notin \partial(n)$, contradiction. Hence we choose $\overline{\varphi} = \varphi_0 : \ p \longrightarrow \partial(n)$ and clearly this is unique. Conversely, if we have a non-surjective map $p \xrightarrow{\varphi_0} n$, then composing with φ_0 defines a natural map $sk_m(p) \longrightarrow \partial(n)$. All told

$$\begin{array}{c} p \longrightarrow cosk_m(\partial(n)) \\ \hline p \longrightarrow \partial(n) \end{array}$$

and $cosk_m(\partial(n)) = \partial(n)$. In other words, $\partial(n)$ is a sheaf for levels $m \geq n$.

Given any $B_n \in \mathcal{E}$, $n \geq 1$, we can "collapse" the subobject $sk_{n-1}(B_n) \hookrightarrow B_n$ to a point via the pushout

$$\begin{tikzcd} sk_{n-1}(B_n) \arrow[r, hook] \arrow[d] & B_n \arrow[d] \\ 1 \arrow[r, hook] & S_n \end{tikzcd}$$

The object S_n has one $k - ball$ in dimension $k < n$; two $n - balls$, one which is non-degenerate; and only their degeneracies in higher dimensions. If we label the degenerate and nondegenerate $n - balls$ (and their degeneracies) by $*$ and 0 respectively, then we can define a commutative monoid structure on S_n by constructing the natural map $S_n \times S_n \longrightarrow S_n$, defined at stages $k \geq n$ by

$$(*,*) \longmapsto *$$

$$(*,0) \longmapsto 0$$

$$(0,*) \longmapsto 0$$

$$(0,0) \longmapsto 0$$

Observe that S_n is an example of a space where a nondegenerate ball can have a degenerate hemisphere. For $n \geq 1$, the unique map $S_n \longrightarrow S_{n+1}$ is constantly the point of S_{n+1}. Of course, S_n is $n - skeletal$, but it is not $n - coskeletal$. Both S^n and S_n are connected for $n \geq 1$ and moreover, S_n (having the structure of a monoid with 0) has the property that ${S_n}^X$ is connected for all $X \in \mathcal{E}$, i.e., S_n is contractible. In section 7 we prove the surprising result that S^n is contractible for $n > 0$.

Now let us see some of the consequences of the Aufhebung and coAufhebung relations.

Consider the diagram below

$$\begin{tikzcd}[row sep=large]
\mathcal{E} \arrow[d, shift right=4.5, "\pi_n \dashv n_! \dashv n^* \dashv n_*"'] \arrow[d, leftarrow, shift right=1.5] \arrow[d, shift left=1.5] \arrow[d, leftarrow, shift left=4.5] \\
\mathcal{E}_n \arrow[d, leftarrow, shift right=3, "i_! \dashv i^* \dashv i_*"'] \arrow[d] \arrow[d, leftarrow, shift left=3] \\
\mathcal{E}_{n-1} \arrow[d, shift right=4.5, "\Gamma_! \dashv \Gamma^* \dashv \Gamma_* \dashv \Gamma^!"'] \arrow[d, leftarrow, shift right=1.5] \arrow[d, shift left=1.5] \arrow[d, leftarrow, shift left=4.5] \\
\mathcal{S}
\end{tikzcd}$$

If $\Gamma_!$ denotes the connected components functor for both $\mathcal{E}_n$ and $\mathcal{E}$, and if Γ^* denotes the discrete inclusion for sets into both $\mathcal{E}_n$ and $\mathcal{E}$, then, for any $X \in \mathcal{E}$, $Y \in \mathcal{E}_n$, $Z \in \mathcal{E}_{n-1}$, and $S \in \mathcal{S}$, we have

$$\begin{array}{c}
i^*\pi_n(X) \longrightarrow Z \\ \hline
\pi_n(X) \longrightarrow i_*(Z) \\ \hline
X \longrightarrow n_! i_*(Z) = n_* i_*(Z) \\ \hline
X \longrightarrow (n-1)_*(Z) \\ \hline
(n-1)^*(X) \longrightarrow Z
\end{array}$$

$$\begin{array}{c}
\Gamma_! n^*(X) \longrightarrow S \\ \hline
n^*(X) \longrightarrow \Gamma^*(S) \\ \hline
X \longrightarrow n_*\Gamma^*(S) = \Gamma^*(S) \\ \hline
\Gamma_!(X) \longrightarrow S
\end{array}$$

Hence $i^*\pi_n = (n-1)^*$ and $\Gamma_! n^* = \Gamma_!$. We therefore know that $\pi_n(X)(B_k) = X(B_k)$ for $k < n$, and so we calculate $\pi_n(X)(B_n)$.

The functor π_n is the left Kan extension of $\mathbf{B} \xrightarrow{p_n} \mathbf{B}_n$; hence, for any $X \in \mathcal{E}$ and $B_n \in \mathbf{B}_n$, $\pi_n(X)(B_n) = X \otimes_{\mathbf{B}} \mathbf{B}(B_n, p_n(-))$. Recall that this is the quotient $\sum_s X(B_s) \times \mathbf{B}(B_n, p_n(B_s))/\sim$ where we "mod-out" by the equivalence relation generated by the identifications $(x, p_n(f)g) = (x \cdot f, g)$. Let $x \otimes k$ be an element of $\pi_n(X)(B_n)$, where $x \in X(B_s), B_n \xrightarrow{k} p_n(B_s)$. If $s \geq n$, then $p_n(B_s) = B_n$, hence $x \otimes k = x \otimes p_n(\delta_i)k = x \cdot \delta_i \otimes k$, where $B_n \overset{\delta_i}{\hookrightarrow} B_s$. This means our analysis reduces to considering only those elements $x \otimes k$, $x \in X(B_n)$, $B_n \xrightarrow{k} B_n$, and since if $y \in X(B_{n+1})$, $y \cdot \delta_0 \otimes 1_{B_n} = y \cdot \delta_1 \otimes 1_{B_n}$, we see that $\pi_n(X)(B_n)$ is the reflexive coequalizer $X(B_{n+1}) \; \substack{\xrightarrow{\delta_0} \\ \xleftarrow{p} \\ \xrightarrow[\delta_1]{}} \; X(B_n) \twoheadrightarrow \pi_n(X)(B_n)$. Consider the diagram below

$$\begin{tikzcd}
 & X(B_{n+1}) \arrow[d, shift left=2] \arrow[d, shift right=2] \\
 & X(B_n) \arrow[u] \arrow[dl, shift left=2] \arrow[dl, shift right=2] \arrow[dd] \\
X(B_{n-1}) \arrow[ur] \arrow[dr] \arrow[dd] & \\
 & \pi_n(X)(B_n) \arrow[ul, shift left=2] \arrow[ul, shift right=2] \arrow[dl] \\
\pi_{n-1}(X)(B_{n-1}) &
\end{tikzcd}$$

The first thing to observe is that there are two induced epimorphisms with a common section; these are simply the maps

$$\pi_n(X)(B_n) \; \substack{\twoheadrightarrow \\ \hookleftarrow \\ \twoheadrightarrow} \; \pi_n(X)(B_{n-1}) = X(B_{n-1}).$$

The second observation is that these two epimorphisms are "coequalized" by $X(B_{n-1}) \twoheadrightarrow \pi_{n-1}(X)(B_{n-1})$. Hence we have $\pi_n(X)(B_n) \twoheadrightarrow \pi_{n-1}(X)(B_{n-1})$ in sets.

We have seen that $\pi_n(X)(B_n)$ is the object part of a reflexive coequalizer (but also $\pi_n(X)(B_k) = X(B_k)$ for $k < n$). Hence, we have the following

Proposition: The functors π_n preserve finite products for $n \geq 0$.

Corollary: The full subcategory of $n - skeletal$ objects in $\mathcal{E}$ is closed under exponentiation by arbitrary objects of $\mathcal{E}$. In particular, $n_!$ preserves exponentiation.

Corollary: The functors π_n preserve homotopies of maps for $n > 0$.

The functors π_n do not preserve equalizers. For example,

$$sk_n(B_{n+1}) \hookrightarrow B_{n+1} \rightrightarrows B_{n+2}$$

is an equalizer diagram, but $\pi_n(sk_n(B_{n+1})) = n^*(B_{n+1})$ is not the equalizer of $B_n \overset{1}{\underset{1}{\rightrightarrows}} B_n$.

2.5 Nonlinear Homology

In this section we define the nonlinear homology of an $X \in \mathcal{E}$ and investigate a particular quotient topos of $\mathcal{E}$.

Consider the following pullback diagram in Cat

$$
\begin{tikzcd}
\mathbf{B}_{mono} \arrow[r, hook, "i"] \arrow[d, "\rho"'] & \mathbf{B} \arrow[d, "q"] \\
\mathcal{W} \arrow[r, hook, "j"'] & \mathbf{C}
\end{tikzcd}
$$

Here i is the (non-full) inclusion of the subcategory $\mathbf{B}_{mono}$ of $\mathbf{B}$, consisting of only the hemisphere maps of $\mathbf{B}$; $\mathbf{C}$ is the quotient category of $\mathbf{B}$ obtained via q by identifying the hemisphere maps of $\mathbf{B}$ at all stages; ρ is the quotient functor to the natural numbers $\mathcal{W}$ which also identifies each pair of hemisphere maps; and j is the (non-full) inclusion of $\mathcal{W}$ in $\mathbf{C}$ in which each unique monomorphism gets a unique retraction in $\mathbf{C}$.

Now consider the diagram of presheaf toposes on each of these categories and the essential geometric morphisms induced by each functor

$$
\begin{tikzcd}
\mathcal{S}^{\mathbf{B}_{mono}^{op}} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, shift right=2] \arrow[d, hookleftarrow] \arrow[d, shift left=2] & \mathcal{E} \arrow[d, shift right=2] \arrow[d, hookleftarrow] \arrow[d, shift left=2] \\
\mathcal{S}^{\mathcal{W}^{op}} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathcal{C}
\end{tikzcd}
$$

Here for simplicity, we write $\mathcal{C}$ for $\mathcal{S}^{\mathbf{C}^{op}}$ and $\mathcal{C}_n$ for $\mathcal{S}^{\mathbf{C}_n^{op}}$, where $\mathbf{C}_n$ is the truncation of $\mathbf{C}$ at stage n. Note that the toposes $\mathcal{S}^{\mathbf{B}_{mono}^{op}}$, $\mathcal{S}^{\mathcal{W}^{op}}$, and their essential subtoposes $\mathcal{S}^{\mathbf{B}_{n\,mono}^{op}}$, $\mathcal{S}^{\mathcal{W}_n^{op}}$ are all examples of étendues since they have sites consisting of only monomorphisms.

For any X in $\mathcal{S}^{\mathbf{B}_{mono}^{op}}$, $\rho_!(X)_n$ is the coequalizer of the structure maps

$$X(B_{n+1}) \rightrightarrows X(B_n),$$

and $\rho_!(X)_{n+1} \longrightarrow \rho_!(X)_n$ is the induced (not necessarily epimorphic) map of co-equalizers. This functor does not preserve finite products: the truncation of $\mathcal{S}^{\mathbf{B}^{op}_{mono}}$ to level one is the topos of irreflexive graphs, where B_1 is the generic edge which also satisfies the equation ${B_1}^2 = B_1 + 2B_0$; this equation is also true in $\mathcal{S}^{\mathbf{B}^{op}_{mono}}$ since $1_* : \mathcal{S}^{\mathbf{B}^{op}_{1\,mono}} \hookrightarrow \mathcal{S}^{\mathbf{B}^{op}_{mono}}$ preserves coproducts and B_0, B_1. In $\mathcal{S}^{\mathcal{W}^{op}}$, the equation $W_n^2 = W_n \ \forall n \in \mathbf{N}$ is true for a trivial reason, and so if $\rho_!$ did preserve products, then $\rho_!(B_1^2) = W_1 + 2W_0$ would give a contraddiction. However, for any $Y \in \mathcal{E}$, $\rho_! i^*(X)_n = \pi_n(X)(B_n)$, and the map induced by $W_n < W_{n+1}$ in $\mathcal{W}$ is $\pi_{n+1}(X)(B_{n+1}) \longrightarrow \pi_n(X)(B_n)$, mentioned before. In particular, $\rho_! i^*$ preserves finite products since the π_n functors do. Note that $\mathcal{S}^{\mathbf{B}^{op}_{mono}}$, like irreflexive graphs, has a π_0 functor which does not preserve finite products and hence is not a "topos of spaces". The opposite is true of $\mathcal{S}^{\mathcal{W}^{op}}$. Here π_0 preserves finite products since, for example, $W_m \times W_n = W_{min\ (m,n)}$. However, in contrast with irreflexive graphs and $\mathcal{S}^{\mathbf{B}^{op}_{mono}}$, the subobject classifier for $\mathcal{S}^{\mathcal{W}^{op}}$ is not connected.

For any $W_n \in \mathcal{W}$, $\rho^*(W_n)$ is the presheaf on $\mathbf{B}_{mono}$ with one element at stages B_k, $k \leq n$, and no elements for $k > n$; the induced map $\rho^*(W_n) \longrightarrow \rho^*(W_{n+1})$ is the obvious inclusion. For any $Y \in \mathcal{S}^{\mathbf{B}^{op}_{mono}}$, a map $\rho^*(W_n) \longrightarrow Y$ picks an element y_n of Y at stage B_n with the property that $y_n \cdot \delta_0 = y_n \cdot \delta_1 = y_{n-1}$, say, and furthermore $y_{n-1} \cdot \delta_0 = y_{n-1} \cdot \delta_1$, etc. Hence a $W_n - element$ of $\rho_*(X)$ is a y_n and the action of $W_{n-1} < W_n$ on y_n is $y_n \longmapsto y_{n-1}$. This means (the counit) $\rho_*(Y) \hookrightarrow Y$ is the subobject of Y consisting of elements whose hemispheres are equal at all stages. We shall call these elements "pancakes".

Since $\mathcal{S}^{\mathcal{W}^{op}}$ is fully included in $\mathcal{S}^{\mathbf{B}^{op}_{mono}}$, there is a canonical map $\rho_* \longrightarrow \rho_!$ and hence a map $\rho_* i^* \xrightarrow{\Phi} \rho_! i^*$. Since the objects "representing" ρ_* (described above) clearly are connected, both $\rho_* i^*$ and $\rho_! i^*$ preserve finite sums and products. Let $\mathcal{E} \xrightarrow{H_*} \mathcal{S}^{\mathcal{W}^{op}}$ be the image of Φ. Then H_* also preserves finite sums and products. For any $X \in \mathcal{E}$

we define the nonlinear homology of X to be $H_*(X)$

$$\begin{tikzcd} \rho_* i^*(X) \arrow[rr, "\Phi_X"] \arrow[dr] & & \rho_! i^*(X) \arrow[dl] \\ & H_*(X) & \end{tikzcd}$$

The elements of $H_n(X)$ are equivalence classes of "pancakes" in X. For a pancake $B_n \xrightarrow{y} X$, if $B_{n+1} \xrightarrow{w} X$ connects y to $x = w \cdot \delta_1$, then

$$x \cdot \delta_i = (w \cdot \delta_1) \cdot \delta_i = w \cdot (\delta_1 \circ \delta_i) = w \cdot (\delta_\circ \circ \delta_i) = (w \cdot \delta_\circ) \cdot \delta_i = y \cdot \delta_i$$

Hence x is also a pancake. This means that a homology class containing a pancake consists only of pancakes. Furthermore, saying that X consists entirely of pancakes is equivalent to saying that no pair of distinct $n - balls$ can be connected. Hence X consists of pancakes if and only if $H_n(X) = X_n \;\; \forall n$.

The functor q^* is the full inclusion of $\mathcal{C}$ in $\mathcal{E}$, the subcategory whose objects consist entirely of pancakes (note that $\rho_* i^* = j^* q_*$ by Beck-Chevalley). For any $X \in \mathcal{E}$, $q_*(X) \hookrightarrow X$ is the subobject of pancakes in X; e.g., $\Gamma^*(S) \hookrightarrow \Gamma^!(S)$ for any $S \in \mathcal{S}$. This category $\mathcal{C}$ is, of course, a quotient topos of $\mathcal{E}$. The essential subtopos $\mathcal{C}_1$ of $\mathcal{C}$ is $\mathcal{S}^{\{0,1\}^{op}}$, the right actions of the commutative two element monoid, studied by Lawvere in [3].

More generally, let $\mathbf{C}_n \xrightarrow{\mu_n} \mathbf{C}$ denote the full inclusion. This functor has a left adjoint

$\mathbf{C} \xrightarrow{k_n} \mathbf{C}_n$ (c.f. $\mathbf{B} \underset{p_n}{\overset{i_n}{\rightleftarrows}} \mathbf{B}_n$) and these induce functors

$$\begin{array}{ccccc} k_{n!} & \dashv k_n^* & \dashv k_{n*} & & \mathcal{C} \\ & \| & \| & & \downarrow\uparrow\downarrow\uparrow \\ \mu_{n!} & \dashv \mu_n^* & \dashv \mu_{n*} & & \mathcal{C}_n \end{array}$$

at the level of presheaf toposes. For any n and $m \geq n$, the maps $C_{m+1} \underset{p}{\overset{\delta}{\rightleftarrows}} C_m$ induce $\mu_n^*(C_{m+1}) \rightleftarrows \mu_n^*(C_m)$ in $\mathcal{C}$. For any $C_k \xrightarrow{f} C_{m+1}$, $k \leq n$ we have $\delta \circ p \circ f = f$, and therefore $\mu_n^*(C_m) \cong \mu_n^*(C_{m+1})$. This means $\mu_{n_*} = \mu_{n_!}$ and $k_{n_!} = k_{n_*}$. Letting $n = 0$, we see that the points and connected components functors of $\mathcal{C}$ are equal (this is also true for $\mathcal{C}_n$). Since $\Omega_{\mathcal{C}}$ has two points, we conclude that it is not connected (similarly, $\Omega_{\mathcal{C}_n}$ is not connected).

Lawvere has shown that the generic object $C_1 = I_{\{0,1\}}$ of $\mathcal{C}_1$ has the property $I^I = 2I^2$. Since the sheaf inclusion μ_{1*} preserves exponentiation, products, and (being also left adjoint to μ_1^*) sums, this equation is also true in $\mathcal{C}$ (and $\mathcal{C}_n$, $n \geq 1$).

However, $q^*(C_1) = S_1$ and since S_1 is contractible (equivalently, $S_1^{S_1}$ is connected) we see that the inclusion of pancakes does not preserve exponentiation.

For $n \neq 1$, $q^*(C_n) \neq S_n$, but for any n it is the $n - skeleton$ of a certain object L that will now prove to be of importance.

In [4] Lawvere exhibits the following relationship between irreflexive graphs $\mathcal{S}^{\bullet \rightrightarrows \bullet} = \mathcal{S}^{\mathbf{B}_{1mono}^{op}}$ and reflexive graphs $\mathcal{S}^{\mathbf{B}_1^{op}}$. For any object $X \in \mathcal{S}^{\mathbf{B}_1^{op}}$, one can construct the full subcategory of $\mathcal{S}^{\mathbf{B}_1^{op}}/X \approx \mathcal{S}^{(\mathbf{B}_1/X)^{op}}$, consisting of those $Y \xrightarrow{f} X$ with discrete fibers:

$$
\begin{tikzcd}
I \arrow[r, "x"] \arrow[d] & Y \arrow[d, "f"] \\
1 \arrow[r, "y"'] \arrow[ur, dashed] & X
\end{tikzcd}
$$

In other words, any edge x in Y that becomes degenerate via f was already degenerate.

These subcategories are all toposes. Let us denote them by $S(X)$. If we specialize to the loop $L = \circlearrowleft\!\bullet$ of $\mathcal{S}^{\mathbf{B}_1^{op}}$, we have the

Proposition (Lawvere): $\mathcal{S}^{\bullet \rightrightarrows \bullet} \simeq S(L)$

Consider now the equivalent construction in $\mathcal{E}/X$, where $X \in \mathcal{E}$. We consider the subcategory of objects $Y \xrightarrow{f} X$ in $\mathcal{E}/X$ with discrete fibers:

$$
\begin{tikzcd}
B_m \arrow[r, "x"] \arrow[d] & Y \arrow[d, "f"] \\
B_n \arrow[r, "y"'] \arrow[ur, dashed] & X
\end{tikzcd}
$$

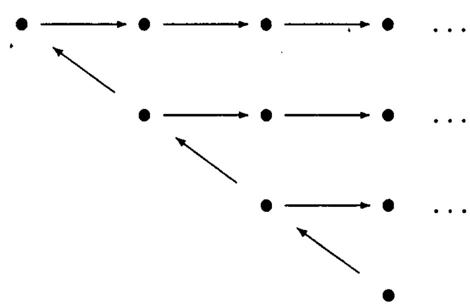
That is, any $m - ball$ x in Y whose image under f is degenerated from an $n - ball$ in X is degenerate itself. Equivalently,

$$
\begin{tikzcd}
sk_n(Y) \arrow[r, hook] \arrow[d, "sk_n(f)"'] & Y \arrow[d, "f"] \\
sk_n(X) \arrow[r, hook] & X
\end{tikzcd}
$$

is a pullback $\forall n \in \mathbf{N}$.

We look for an object L in $\mathcal{E}$ which will play the role of the loop in reflexive graphs.

Let L be the ball complex sketched in part below



This object has precisely one nondegenerate and n degenerate $n-balls$, the latter collection having one $n-ball$ for each possible degeneration. Hence $E \longrightarrow L$ belongs to $S(L) \subset \mathcal{E}/X$ if and only if nondegenerates are mapped to nondegenerates, and $m-balls$ degenerated from nondegenerate $n-balls$ are mapped to the unique $m-ball$ in L parameterizing these elements. Thus we have the

Proposition: $S(L) = \mathcal{S}^{\mathbf{B}^{op}_{mono}}$.

Corollary: Let $L_n = i_n^*(L) = q^*(C_n)$. If $S_n(X)$ is the analogous construction in $\mathcal{E}_n/X$ for $X \in \mathcal{E}_n$, then $S_n(L_n) = \mathcal{S}^{\mathbf{B}^{op}_{n\,mono}}$.

2.6 Abelian Group Objects in $\mathcal{E}$

We now look at the category $\mathcal{A}$ of Abelian groups in the topos of Ball Complexes. We shall show that $\mathcal{A}$ is equivalent to the category $\mathcal{K}$ of Abelian chain complexes (c.f. [14]) and we shall also give an example to show that there is not a normalization theorem for $\mathcal{A}$ (c.f. [14]).

Given an object G in $\mathcal{A}$ we define the chain complex $A(G)$ by $A_n(G) = G_n$ and where the differential is defined on G_n by $\partial = \delta_1 - \delta_0$. This is indeed a differential since

$$\partial^2 = \partial \circ \partial = (\delta_1 - \delta_0) \circ (\delta_1 - \delta_0) = \delta_1 \circ \delta_1 - \delta_0 \circ \delta_1 - \delta_1 \circ \delta_0 + \delta_0 \circ \delta_0 = \delta_1 - \delta_0 - \delta_1 + \delta_0 = 0$$

Define $N(G)^2$ to be the subcomplex of $A(G)$ whose nth term is $N_n(G) = ker(G_n \xrightarrow{\delta_0} G_{n-1})$ and whose differential is given by $\partial(x) = \delta_1(x)$, $x \in N_n(G)$. This defines a differential since

$$\partial(\partial(x)) = (x \cdot \delta_1) \cdot \delta_1 = (x \cdot \delta_0) \cdot \delta_1 = 0 \cdot \delta_1 = 0.$$

Furthermore, A and N define functors $\mathcal{A} \longrightarrow \mathcal{K}$.

Examples: For any $X \in \mathcal{E}$ we can define an object $\Lambda(X)$ in $\mathcal{A}$ with $\Lambda(X)_n = \mathbf{Z} \cdot X_n$, the free Abelian group on X_n, and whose hemisphere and degeneracy maps are those uniquely determined by the hemisphere and degeneracy maps of X. If we compose A and Λ

$$\begin{tikzcd}
& \mathcal{E} \arrow[dl, "\Lambda"'] \arrow[dr, "C"] & \\
\mathcal{A} \arrow[rr, "A"'] & & \mathcal{K}
\end{tikzcd}$$

we obtain a functor which we shall denote by $C = A \circ \Lambda$. Let $X = B_n$, $n > 0$. Then $C_n(X) = (2n+1)\mathbf{Z}$ where one generator is nondegenerate and all others are degenerate. $C_{n+1}(X) = (2n+1)\mathbf{Z}$ and all generators are degenerate; hence im $(C_{n+1}(X) \xrightarrow{\partial} C_n(X)) = 0$. $C_{n-1}(X) = 2n\mathbf{Z}$, where two generators (coming from $B_{n-1} \xhookrightarrow{\delta_i} B_n$) are nondegenerate and all others are degenerate. The kernel of $C_n(X) \xrightarrow{\partial} C_{n-1}(X)$ is $2n\mathbf{Z}$ and hence we have $H_n(C(X)) = 2n\mathbf{Z}$. Since none of the elements of $C_n(X)$ are in the kernel of δ_0, we have $N_n(\Lambda(X)) = 0$ and therefore $H_n(N(\Lambda(X))) = 0$. This means that the inclusion of complexes $N(G) \hookrightarrow A(G)$ need not induce an isomorphism on homology for an object $G \in \mathcal{A}$ (c.f. [14]).

[2] $N(G)$ is not an object of $\mathcal{A}$

For any $G \in \mathcal{A}$ we can form the subcomplex $D(G) \hookrightarrow A(G)$ consisting of degenerates of G. The differential on $D(G)$ is the restriction of the differential of $A(G)$ and is identically zero. The quotient complex $C_N(G) = A(G)/D(G)$ with the induced differential is called the normalized complex of G. In the above example $C_{n+1}(X)$ consists entirely of degenerates and therefore im $(C_{N_{n+1}}(\Lambda(X)) \xrightarrow{\partial} (C_{N_n}(\Lambda(X))) = 0$. The element $1_{B_n} \in C_n(X)$ is the only nondegenerate generator, and therefore $C_{N_n}(\Lambda(X)) \xrightarrow{\partial} C_{N_{n-1}}(\Lambda(X))$ is the map $\mathbf{Z} \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$, $1_{B_n} \longmapsto (\delta_1 - \delta_0)$. Clearly the kernel of this map is zero and therefore $H_n(C_N(\Lambda(X))) = 0$. This means that the complexes $C_N(G)$ and $A(G)$ need not be chain equivalent for $G \in \mathcal{A}$ (c.f. [14]).

Theorem: (Dold-Kan-Moore theorem for Ball Complexes). If G is an Abelian group in $\mathcal{E}$, then $A_n(G) = \prod_{i=0}^{n} N_i(G)$, for all n.

Proof: When $n = 0$, the equation is true. Assume the equation is true for $n < q$. We have $G_q \underset{\delta_0}{\overset{p}{\rightleftarrows}} G_{q-1}$ with $\delta_0 p = 1$ and hence $G_q = ker(\delta_0) \times im(p) = N_q(G) \times G_{q-1}$. By the induction hypothesis applied to G_{q-1}, we have

$$G_q = N_q(G) \times \prod_{i=0}^{q-1} N_i(G).$$

We now construct a functor $\Gamma : \mathcal{K} \longrightarrow \mathcal{A}$ which is inverse to N. Given a chain complex K, define $\Gamma(K)$ as follows: $\Gamma_n(K) = \prod_{i=0}^{n} K_i$. The hemisphere maps $\Gamma_n(K) \xrightarrow{\delta_i} \Gamma_{n-1}(K)$ are defined by $\delta_1 = < \pi_0, \ldots, \pi_{n-1} + \partial \circ \pi_n >$, $\delta_0 = < \pi_0, \ldots, \pi_{n-1} >$, and the map $\Gamma_{n-1}(K) \xrightarrow{p} \Gamma_n(K)$ is $p = < 1, 0 >$. Note that $ker(\delta_0) \cong K_n$. $\Gamma(K)$ is an object of $\mathcal{A}$ and Γ is a functor. $N_n(\Gamma(K)) = ker(\Gamma_n(K) \xrightarrow{\delta_0} \Gamma_{n-1}(K)) \cong K_n$, and the differential is that of K. On the other hand, for $G \in \mathcal{A}$, $\Gamma_n(N(G)) = \prod_{i=0}^{n} N_i(G) = G_n$, by the Theorem. The hemisphere and degeneracy maps coincide with those of G. Hence $\Gamma \circ N = 1_{\mathcal{A}}$, $N \circ \Gamma = 1_{\mathcal{K}}$.

2.7 Homotopy Categories Revisited

In section 4 we looked at the homotopy category $\mathcal{H}(\mathcal{E})$, consisting of homotopy classes of maps in $\mathcal{E}$. In fact, there are a countably infinite many categories of this ilk: Given $n \in \mathbf{N}$, define $\mathcal{H}_n(\mathcal{E})$ to be the category whose objects are those of $\mathcal{E}$ and whose "hom-sets" are $\mathcal{H}_n(\mathcal{E})(Y, X) = \pi_n(X^Y)(B_n)$, where $X, Y \in \mathcal{E}$. As before, the product preservation property of π_n ensures that composition can be defined and that indeed $\mathcal{H}_n(\mathcal{E})$ is a category. (For $n > 0$, the map $\pi_n(X)(B_n) \longrightarrow \pi_{n-1}(X)(B_{n-1})$ is the arrows function of a functor $\mathcal{H}_n(\mathcal{E}) \longrightarrow \mathcal{H}_{n-1}(\mathcal{E})$ which is the identity on objects).

For $n > 0$, if $X \in \mathcal{E}$ has the property $\mathcal{H}_n(\mathcal{E})(Y, X) = 1 \ \forall Y \in \mathcal{E}$, i.e., $\pi_n(X^Y)(B_n) = 1$, then in particular $X^Y(B_0) = 1$ and therefore there is precisely one map $Y \longrightarrow X$ for any Y. Hence, we see that $X = 1$. In short, X is terminal in $\mathcal{H}_n(\mathcal{E})$ if and only if it is a terminal object in $\mathcal{E}$. As we have already seen, this is not the case for $n = 0$, and as we shall now see, in $\mathcal{H}_0(\mathcal{E})$ the representables are terminal. Before doing this let us remark that we could have required that $\mathcal{H}_n(\mathcal{E})(Y, X) = \pi_n(X^Y)$, dropping the requirement that the presheaf be evaluated at B_n. In this way the homotopy category is enriched in $\mathcal{E}_n$ as opposed to sets. For example, $\mathcal{H}_1(\mathcal{E})$ is then enriched in reflexive graphs.

Proposition: B_n is a terminal object of $\mathcal{H}_0(\mathcal{E})$.

Proof: The points of $B_n^{B_n}$ are maps $B_n \longrightarrow B_n$. We shall show that $B_n \xrightarrow{1} B_n$ can be connected to the constant map $B_n \twoheadrightarrow B_0 \xhookrightarrow{\delta_0} B_n$ (we could have picked the other constant map). Of course, by "connected to" we mean a homotopy $B_1 \times B_n \xrightarrow{\Phi} B_n$

It is trivial if $n = 0$, so we assume $n > 0$. The elements that "generate" $B_1 \times B_n$ are the $n - balls$ $(\delta_i p, 1_{B_n})$, $(p\prime, 1_{B_n})$ (where $p\prime$ is the map $B_n \twoheadrightarrow B_1$ or the identity if $n = 1$), and the $1-balls$ $(1, \delta_i p)$. Labelling these x_i, y, and a_i respectively, the incidence relations are as follows:

$$y \cdot \delta_i = x_i \cdot \delta_i = a_i \cdot \delta_i$$

$$x_0 \cdot \delta_1 = a_1 \cdot \delta_0$$

$$x_1 \cdot \delta_0 = a_0 \cdot \delta_1$$

. Define Φ by

$x_0 \longmapsto B_n \twoheadrightarrow B_0 \xhookrightarrow{\delta_0} B_n$, $x_1 \longmapsto 1_{B_n}$, $y \longmapsto 1_{B_n}$, $a_0 \longmapsto B_1 \twoheadrightarrow B_0 \xhookrightarrow{\delta_0} B_n$ (any j), $a_1 \longmapsto B_1 \xhookrightarrow{\delta_i} B_n$ (any i).

Let X be a ball complex with only one point. The presheaf $B_1 \times X$ is generated by the $k-balls$ $(\delta_i p, x)$, $(p\prime, x)$ where x is a $k-ball$ in X and $B_k \xrightarrow{p} B_0$, $B_k \xrightarrow{p\prime} B_1$, where $p\prime$ can be the identity. Let $X \underset{\varphi_1}{\overset{\varphi_0}{\rightrightarrows}} X$ be two endomaps of X. Defining $B_1 \times X \xrightarrow{\Phi} X$ by $\Phi(\delta_i p, x) = \varphi_i(x)$, $\Phi(p\prime, x) = x$, we have the

Proposition: If $X \in \mathcal{E}$ has only one point, then it is contractible.

Recall that $\mathcal{C}$ has the property that $\pi_0 = pts.$; hence, for any object X consisting of pancakes, $\pi_0(X) = pts.(X)$. This means an object consisting of pancakes is contractible if and only if it is connected.

The full subcategory of one point objects of $\mathcal{E}$ is a reflective subcategory: Given any $X \in \mathcal{E}$, the reflection $\hat{X}$ is obtained by identifying all the points of X via the pushout

$$\begin{tikzcd}
disc.\ pts.\ (X) \arrow[r, hook, "\in_X"] \arrow[d] & X \arrow[d] \\
1 \arrow[r, hook] & \hat{X}
\end{tikzcd}$$

Since *pts.* preserves pushouts and 1, and since *pts.* $(\in_X)$ is an isomorphism, we have *pts.* $(\hat{X}) = 1$. For example, $\hat{B}_1 = S_1$

Recall now the functor described in the first paragraph of page 1. Viewing this functor as taking values in the category **Top** of topological spaces and continuous maps we can take it its left Kan extension to obtain a realization functor $\mathcal{E} \xrightarrow{r} \mathbf{Top}$. This functor is left adjoint to the principal complex functor $\mathbf{Top} \longrightarrow \mathcal{E}$, assigning to any space $X \in$ **Top** the Ball Complex whose value at any B_n is the set of all continous maps from B_n in $\mathbf{R}^n$ to X. We shall now show that r does not preserve finite products. First we need to return to the $n - sphere$ S^n in $\mathcal{E}$.

S^n can be obtained as a pushout

$$\begin{tikzcd} 2B_{n-1} \arrow[r] \arrow[d] & B_n \arrow[d, "\beta"] \\ B_n \arrow[r, "\alpha"'] & S^n \end{tikzcd}$$

If we apply the realization functor r to this diagram, since r is a left adjoint and hence preserves pushouts, we see that $r(S^n)$ is the $n - sphere$ in $\mathbf{R}^{n+1}$.

Now we need to study the object $S^n \times I$ in $\mathcal{E}$. $S^n \times I$ is generated by the n-balls $a_k = (\delta_k, p\prime)$, $y_{l,s} = (\delta_l,\ \delta_s\ p)$, where $B_n \xhookrightarrow{\delta_k} B_{n+1}$, $B_n \xrightarrow{p'} B_1$ (p' can be 1_{B_1}) and $B_n \xrightarrow{p} B_0$, $n > 0$, and the 1-ball $x_j = (\delta_j p, 1)$, where $B_0 \xhookrightarrow{\delta_j} B_{n+1}$. The incidence relations are

$$a_k \cdot \delta_0 = x_0 \cdot \delta_0 = y_{l,0} \cdot \delta_0,$$

$$a_k \cdot \delta_1 = x_1 \cdot \delta_1 = y_{l,1} \cdot \delta_1,$$

$$x_0 \cdot \delta_1 = y_{l,1} \cdot \delta_0,$$

$$x_1 \cdot \delta_0 = y_{l,0} \cdot \delta_1.$$

We define Φ as follows:

$$a_0 \mapsto \delta_k : B_n \hookrightarrow B_{n+1}, \quad k \ \ arbitrary$$

$$a_1 \mapsto \delta_l : B_n \hookrightarrow B_{n+1}, \quad l \ \ arbitrary$$

$$x_0 \mapsto \delta_0 p : B_1 \to B_{n+1},$$

$$x_1 \mapsto \delta_i : B_1 \hookrightarrow B_{n+1}, \quad i \ \ arbitrary$$

$$y_{0,0} \mapsto \delta_0 p : B_n \to B_{n+1},$$

$$y_{1,0} \mapsto \delta_0 p : B_n \to B_{n+1},$$

$$y_{0,1} \mapsto \delta_0 : B_n \hookrightarrow B_{n+1},$$

$$y_{1,1} \mapsto \delta_1 : B_n \hookrightarrow B_{n+1}.$$

Hence we have the

Proposition S^n is contractible for $n > 0$.

A topological space X is contractible if the identity map 1_X is homotopic to a constant endomap $X \longrightarrow 1 \longrightarrow X$. It is well known that the $n - sphere$ in $\mathbf{R}^{n+1}$ is not a contractible space. Suppose now that the realization functor r preserves finite products. We can apply r to the homotopy of the proposition to obtain a homotopy in **Top** connecting the identity on the $n - sphere$ to a constant. Contradiction.

2.8 The points of $\mathcal{E}$

Diaconescu's Theorem (see [13])tells us that points $\mathcal{S} \longrightarrow \mathcal{S}^{\mathbf{C}^{op}}$ correspond to filtering functors $\mathbf{C} \longrightarrow \mathcal{S}$ for any category $\mathbf{C}$. We first calculate the filtering functors $\mathbf{B} \longrightarrow \mathcal{S}$ and then the inverse image functors of the corresponding points.

A filtering functor $\mathbf{B} \xrightarrow{F} \mathcal{S}$ satisfies the following three conditions:

(1) $F(B_0) \neq \emptyset$

(2) If $a \in F(B_n)$, $B_n \overset{f}{\underset{g}{\rightrightarrows}} B_m$ and $F(f)(a) = F(g)(a)$ then there are $B_k \xrightarrow{h} B_n$ and $b \in F(B_k)$ with $fh = gh$ and $F(h)(b) = a$

(3) If $a \in F(B_n)$, $b \in F(B_m)$ then there are $B_k \xrightarrow{f} B_n$, $B_k \xrightarrow{g} B_m$ and $c \in F(B_k)$ with $F(f)(c) = a$, $F(g)(c) = b$.

For any filtering F, $F(B_0) = \{*\}$, the one element set: If $a \neq b$ are elements of $F(B_0)$, then (3) implies there are $B_k \overset{f}{\underset{g}{\rightrightarrows}} B_0$, $c \in F(B_k)$ such that $F(f)(c) = a$, $F(g)(c) = b$. Since $B_k \longrightarrow B_0$ is unique we see that this cannot be true.

For any l, $B_0 \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} B_l$ we have $F(\delta_0)(*) \neq F(\delta_1)(*)$: If not, then (2) implies there is a map $B_k \xrightarrow{h} B_0$ such that $\delta_0 h = \delta_1 h$. It is clear that no h can have this property.

In particular, $F(B_1)$ has at least two elements. It can have three elements but no more for the following reason: If a, b are distinct elements of $F(B_1)$ not in the image of either $F(\delta_0)$ or $F(\delta_1)$, then by (3) we have $B_k \overset{f}{\underset{g}{\rightrightarrows}} B_1$, $c \in F(B_k)$ and $F(f)(c) = a$, F(g)(c)=b. Clearly $k > 0$, and $f \neq g$ implies at least one of these must factor through some $B_0 \xhookrightarrow{\delta_i} B_1$. Therefore, a or b is in the image of $F(\delta_i)$. Contradiction.

If $a \in F(B_1)$ is not in the image of $F(\delta_i)$, then $F(\delta_0)(a) \neq F(\delta_1)(a)$: If not, then by (2) we have $b \in F(B_k)$, $B_k \xrightarrow{h} B_1$, $F(h)(b) = a$, and $\delta_0 h = \delta_1 h$. This cannot happen by a previous analysis.

We may say that $a \in F(B_l)$, $l > 0$ is an adjoined element if it is not in the image of an $F(\delta_i)$. If x and y are adjoined elements, then $F(p)(y) = x$ (assuming "$y > x$"): For if x, y are in $F(B_n)$, $F(B_m)$ respectively, then by (3) we have an element $z \in F(B_k)$, $B_k \xrightarrow{f} B_n$, $B_k \xrightarrow{g} B_m$ with $F(f)(z) = x$, $F(g)(z) = y$. Since x, y are not in the image of any $F(\delta_i)$, we must have f and g both epimorphisms and therefore $F(p)(y) = x$.

Using the above analysis we see that, for an arbitrary n, $F(B_n)$ has at most $2n + 1$ elements.

Suppose that $F \in Filt\,(\mathbf{B})$ and $\Lambda \subseteq \mathbf{N}$ is the subset (which includes zero) indexing the stages in which there is an adjoined element. The inverse image of the point $\mathcal{S} \xrightarrow{f} \mathcal{E}$ corresponding to F is $f^* = \_\otimes_{\mathbf{B}} F$. We shall now calculate $f^*(X)$ for an arbitrary $X \in \mathcal{E}$. Recall that

$$X \otimes_{\mathbf{B}} F = \sum_{\mathbf{N}} X(B_s) \times F(B_s)/\sim$$

where $\sim$ is the equivalence relation generated by the identifications $(x \cdot \alpha,\, y) = (x,\, \alpha \cdot y)$, $x \in X(B_s)$, $y \in F(B_t)$, $B_t \xrightarrow{\alpha} B_s$. If $x \in X(B_r)$, $z \in F(B_r)$ the equivalence class of (x, z) is denoted by $x \otimes z$. Given any such $x \otimes z$ we can reduce this (via the above relations) to a unique $x' \otimes z'$ with $x' \in X(B_k)$ nondegenerate, $z' \in F(B_k)$ an adjoined element, and $k \in \Lambda$: For if $x = x' \cdot p$ with x' nondegenerate, and $z = \delta_i(z')$ with z' adjoined, then

$$x \otimes z = x'p \otimes z = x' \otimes p \cdot z = x' \otimes p\delta_i \cdot z'.$$

If $p\delta_i$ is δ_k for some k, then

$$x' \otimes p\delta_i \cdot z' = x' \otimes \delta_k \cdot z' = x' \cdot \delta_k \otimes z'.$$

If $x' \cdot \delta_k$ is degenerate, say $x' \cdot \delta_k = w \cdot p$ for some nondegenerate w, then $x' \cdot \delta_k \otimes z' = w \cdot p \otimes z' = w \otimes p \cdot z'$ and $p \cdot z'$ is an adjoined element. If $p \cdot \delta_i$ is an epimorphism, then $p\delta_i \cdot z'$ is an adjoined element. The uniqueness of x', z' is easy. Thus we have

$$X \otimes_{\mathbf{B}} F \cong \bigcup_{\lambda \in \Lambda} \textit{nondeg. } X(B_\lambda).$$

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