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CLOSED BICATEGORIES AND VARIABLE CATEGORY THEORY

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Author's note. We show that many notions relative to locally internal categories over a topos **E** are standard notions of enriched category theory, provided the enrichment is taken in the bicategory Span**E**. The appropriate properties of Span**E** give the formal notion of closed bicategory. Furthermore a common setting for internal categories and locally internal categories is obtained.

This work is an extended version of a paper in preparation with the same title.

Sunto. In questo lavoro si mostra che numerose nozioni relative a categorie localmente interne ad un topos **E** diventano nozioni standard della teoria delle categorie arricchite, pur di assumere come base la bicategoria Span**E**. Le proprietà di Span**E**, opportunamente astratte, forniscono la nozione di *bicategoria chiusa*. Si ha in tal modo un ambiente comune sia per le categorie interne che per quelle localmente interne.

Questo lavoro raccoglie i seminari tenuti dagli autori durante i mesi di settembre e ottobre 1983 al Sydney Category Seminar, e costituisce una versione estesa di un lavoro in preparazione con lo stesso titolo.

Commentary by R. Betti:

BASE BICATEGORIES

The following paper: CLOSED BICATEGORIES AND VARIABLE CATEGORY THEORY, written with Robert F.C. Walters, was published as a report of Milan Department of Mathematics (Quaderno 5/1985). It is based on the notion of categories enriched in a bicategory and reports a series of talks given by the authors at the Sydney Category Seminar during September and October 1983. Its essential aim is to show that the notion is suitable to give a common setting for internal and locally internal category theory relative to a given base topos (what we called "variable category theory") provided the enrichment is taken in the bicategory of the Spans of the topos, and moreover to analyze the properties of the base bicategory which are necessary to develop further the theory (the notion of a closed bicategory).

The idea of enriching a category in a bicategory first arose from the attempt of generalizing the categorical structure of classical automata to the case of tree automata. In [1] Una teoria categoriale degli automi, 1979 (A categorical theory of automata) and [2]

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Automi e categorie chiuse, 1980 (Automata and closed categories) classical automata are regarded as categories enriched in a monoidal closed category obtained by the free monoid of inputs. A monoidal category is just a one-object bicategory while a "variable monoidal category" was necessary for the generalization to tree automata, viewed as "many sorted" classical automata. In this case the necessary base bicategory is described in the subsequent paper [8] written with S. Kasangian: Tree automata and enriched category theory.

The main motivation for all this work is to be found in the seminal paper [9] by F.W. Lawvere, Metric spaces, generalized logic, and closed categories, more precisely in the thesis that while "it is a banality that all the mathematical structures of a given kind constitute the objects of a category" it is true that "fundamental structures are themselves categories".

The possibility of enriching in a bicategory was soon communicated (by ordinary mail) to my Milan colleagues Aurelio Carboni, visiting at that time (1979-80) Bill Lawvere at Buffalo, and Stefano Kasangian, visiting G. Max Kelly at Sydney, giving rise to a strong collaboration between the category groups at Milan University in Italy and at Sydney and Macquarie Universities in Australia.

The new enrichment first appeared, in Italian, in a series of reports of the Mathematical Institute of Milan University, i.e. [3] Bicategorie di base, 1981 (Base bicategories), [4] Alcune proprietà delle categorie basate su una bicategoria, 1982 (Some properties of categories based on a bicategory) and others, which extend some results of [9] to locally preordered bicategories.

Soon it was clear that the new enrichment was suitable to describe more situations, relative to categories whose homs can be thought to live in a "variable" monoidal base. In a short time, the papers by R. Betti and A. Carboni [5,6] on an intrinsic notion of topology and by R.F.C. Walters [10, 11] on the associated sheaf regarded as a Cauchy complete category were obtained. The general aspects of the theory were later incorporated in R. Betti, A. Carboni, R. Street and R.F.C. Walters [7] Variation through enrichment.

I take the opportunity to recall here the warm and active collaboration with Bob Walters and Aurelio Carboni, friends who unfortunately passed away a few years ago.

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Introduction

Our point of view is that variable categories are categories enriched over a variable base. In this paper we are attempting two things: (i) to analyse the properties of a base bicategory which are necessary to develop category theory enriched in that base, and (ii) to develop the particular example of locally internal categories over a topos.

Usually, a locally internal category is thought of as a fibration over a given topos $\mathbf{E}$ which provides the domain of variation (Lawvere [19], Penon [22], Bénabou [1], Paré and Schumacher [21], Street [24]). In Penon's formulation the fibre over $u \in \mathbf{E}$ is enriched over $\mathbf{E}/u$. Our description is two-sided (as implicitly indicated in Lawvere [19]); we regard a locally internal category as being enriched in Span$\mathbf{E}$. In place of the category $\mathbf{E}$ of parameters, we thus have a bicategory of parameters, Span$\mathbf{E}$, and the theory of variable categories can be developed as category theory enriched in a bicategory. The appropriate properties of the base bicategory, as abstracted by Span$\mathbf{E}$, give the general notion of closed bicategory. We have chosen not to describe all the results at this level of generality: in some cases we give just particular examples.

Many of the important notions of locally internal category theory are exactly standard notions of enriched category theory. For example the universal property of cartesian arrows and completeness (with Beck-Chevalley condition) become cases of completeness in the sense of indexed limits. Functor categories can be defined as usual in enriched category theory, using ends.

Furthermore our approach provides a common setting for internal categories and locally internal categories. An important feature in our development of enriched category theory is that we do not, as is usual, use external completeness of the base. We use only elementary properties of the base, in particular completeness with respect to internal categories. Clearly most of the results hold with the assumption of external completeness.

The theory of categories enriched in a base bicategory first arose in Betti [3], Walters [28]. The subject has been developed by Betti, Carboni, Kasangian, Street and Walters (see references). The important notion of a tensor product on a bicategory used in this paper was introduced by Carboni and Walters [12].

This work is part of a collaboration which has been made possible by the Italian CNR and Sydney and Macquarie Universities. It reports the talks given by the authors at the Sydney Category Seminar, during September-October 1983. A joint paper with the same title is in preparation. We thank members of the Sydney Category Seminar for helpful discussions.

1 Locally internal categories as enriched categories

1.1 Question *What should a variable category be?*

We think of variable categories as having objects $x, y, \ldots$ parametrized by (variable) sets $u, v, \ldots$ and arrows $hom(x, y)$ parametrized by the product $u \times v$. More formally, the basic example of a variable category is given by Fam$\mathbf{C}$, the category of families of objects of $\mathbf{C}$ indexed by small sets, $\mathbf{C}$ being a locally small category. If $x = (x_i)_{i\in u}$ and $y = (y_j)_{j\in v}$ are two objects of Fam$\mathbf{C}$, then $hom(x, y)_{i,j} = \mathbf{C}(x_i, y_j)$ where $(i, j) \in u \times v$.

A second way of regarding Fam$\mathbf{C}$ is that it is a category fibered over Sets: the object $x = (x_i)_{i\in u}$ lies over u, and arrows from $x = (x_i)$ to $y = (y_j)$ in the total category are pairs (f, α_i), where f is a change of parameter $f : u \to v$ and $\alpha_i : x_i \to y_{fi}$ is a family of maps of $\mathbf{C}$. Composition and identities are defined in an obvious way. It is known that the notion of fibration contains the information needed to replace Sets in the example by an arbitrary topos $\mathbf{E}$ (see Bénabou [2], Lawvere [19]).

We will develop the first point of view toward Fam$\mathbf{C}$; namely that it is a category enriched over Span(Sets). The enrichment over Span(Sets) is an instance of the following general:

1.2 Definition *Let B denote a bicategory. A* category based on B (or a B-category) *X consists of:*
(i) objects $x, y, \dots$
(ii) an underlying function which assigns to any object x an object ex in B,
(iii) for each pair of objects a hom, i.e. an arrow in B

$$X(x, y) : ex \to ey$$

(iv) for each object a unit, i.e a 2-cell $1_{ex} \to X(x, x)$,
(v) a composition, *i.e a 2-cell associated to every triple of objects*

$$X(y, z) \cdot X(x, y) \to X(x, z)$$

All these data are required to satisfy the associativity and the identity laws.

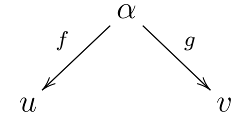
1.3 Definition *When X and Y are B-categories, a B-*functor *$F : X \longrightarrow Y$ is a function on objects which preserves the underlying object, and a family of 2-cells (which express the effect of the functor on arrows)*

$$X(x, y) \to Y(Fx, Fy)$$

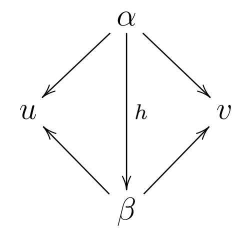
These data are required to satisfy usual axioms for functors.

In the case when B is a symmetric monoidal category considered as a bicategory with one object, we get the usual notion of categories enriched in B. For the case of a general bicategory B, see the references (for instance [8]). In this paper the main example of a base bicategory is Span$\mathbf{E}$ where $\mathbf{E}$ is an elementary topos.

1.4 Example (Span$\mathbf{E}$). Objects of Span$\mathbf{E}$ are objects of $\mathbf{E}$, arrows $\alpha : u \longrightarrow\!\!\!\!\!\!\!+\ \ v$ are spans (f, g) of maps in $\mathbf{E}$ as in the picture:



2-cells $\alpha \to \beta$ are maps h in $\mathbf{E}$ such that the following triangles commute



Composition is given by pullback and $\Delta_u = (1_u, 1_u)$ is the identity.

Span$\mathbf{E}$ is a *symmetric* bicategory, in the sense that to each arrow $\phi = (f, g)$ is associated an *opposite arrow* $\phi^\circ = (g, f)$ with the properties:

$$\Delta_u^\circ \cong \Delta_u$$
$$(\psi \cdot \phi)^\circ \cong \phi^\circ \cdot \psi^\circ$$
$$(\phi^\circ)^\circ \cong \phi$$

Moreover any 2-cell $\phi \to \psi$ corresponds exactly to one 2-cell $\phi^\circ \to \psi^\circ$.

A map f of $\mathbf{E}$ becomes the arrow $(1, f)$ and such arrows are characterized (up to isomorphism) by the fact that they have right adjoints (a right adjoint of f is f°). In a general bicategory B we call an arrow which has a right adjoint a *map*.

Maps of $\mathbf{E}$ will be called simply maps when considered in Span$\mathbf{E}$.

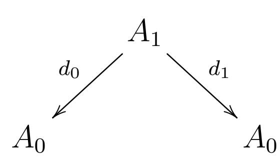
A useful way of regarding Span$\mathbf{E}$ is to consider arrows as matrices $(\alpha_{ij})_{i \in u, j \in v}$ of objects of $\mathbf{E}$. Then composition is matrix product

$$(\beta \cdot \alpha)_{ik} = \sum_j \beta_{jk} \times \alpha_{ij}$$

1.5 Example (Families). It is now easy to see that Fam$\mathbf{C}$ can be regarded as a Span$\mathbf{E}$-category ($\mathbf{E}$ = Sets), by taking the same objects and by defining the *hom* from (x_i) to (y_j) to be the matrix $\mathbf{C}(x_i, y_j)_{i \in u, j \in v}$.

From now on we will denote this *hom* by Fam$\mathbf{C}(x, y)$. Notice that this does not mean the set of arrows from (x_i) to (y_j) in Fam$\mathbf{C}$ regarded as the total category of a fibration.

1.6 Example (Internal categories). If A is a category internal to $\mathbf{E}$, then it becomes an arrow $A_0 \longrightarrow\!\!\!\!\!\!\!+\;\; A_0$



with a monad structure in Span$\mathbf{E}$. Thus an internal category is exactly a Span$\mathbf{E}$-category with only one object, whose underlying object is A_0 and whose *hom* is (d_0, d_1).

It should be noted that functors between internal categories are not the same as Span**E**-functors, but rather are mappings between monads. More precisely a functor $A \to D$ between internal categories amounts to a map $F : A_0 \to D_0$ and a 2-cell $A_1 \to f^\circ \cdot D_1 \cdot f$ compatible with compositions and identities in A and D.

Later we will see that there is a natural way of representing functors between internal categories as actual Span**E**-functors.

1.7 Definition *A B-category with one object will be called an* internal category.

Any object u of the base provides an example of an internal category with the *hom* equal to 1_u. Such internal categories we call *discrete*.

Recall that Fam**C** is both a fibration and a Span(Sets)-category. We can now show that the universal property of fibrations can be substituted by the notion of *restriction*.

1.8 Definition *Let X be a B-category, x an object over u and $f : v \to u$ a map. A* restriction *x_f of x along f is an object over v such that the following* restriction laws

$$X(x, y) \cdot f \cong X(x_f, y)$$
$$f^\circ \cdot X(y, x) \cong X(y, x_f)$$

hold for each object y. We say that X has restrictions when for each x and each map f a restriction x_f exists.

1.9 Remark We will see later that, under usual conditions for B (satisfied by Span**E**), the two properties of the above definition are equivalent, i.e. each implies the other.

1.10 Example (Families). Reconsider the example of Fam**C** as a fibration. If $f : v \to u$ is a map, then any object $(x_i)_{i\in u}$ can be pulled back to the object $f^*(x_i)_{i\in u} = (x_{fj})_{j\in v}$. Regarding Fam**C** as a Span(Sets)-category, the same object satisfies the properties required by restrictions. Indeed we have:

$$\mathrm{Fam}\mathbf{C}(f^*x, y)_{jk} \cong \sum_i \mathrm{Fam}\mathbf{C}(x, y)_{jk} \times f_{ij}$$

where

$$f_{ij} = \begin{cases} * \text{ if } fj = i \\ \phi \ \text{ otherwise} \end{cases}$$

Let us consider a fibration $F \xrightarrow{p} \mathbf{E}$, with F locally small. For any pair of objects x, y we have a functor

$$\mathrm{Span}\mathbf{E}(px, py)^{\mathrm{op}} \xrightarrow{[x,y]} \mathrm{Sets}$$

which takes (f, g) into $F_w(f^*x, g^*y)$, where w is the common domain of f and g, and F_w denotes the fibre over 1_w in F.

In the main example this functor is represented by the matrix $\mathbf{C}(x_i, y_j)$, $i, j \in px \times py$. We are thus led to the:

1.11 Definition *A* locally internal category F *over* $\mathbf{E}$ *is a fibration*

$$p : F \longrightarrow \mathbf{E}$$

such that the functor $[x,y] : \mathrm{Span}\mathbf{E}(px,py)^{\mathrm{op}} \to \mathrm{Sets}$ *is representable, for each pair* x,y.

1.12 Remark This notion is a two-sided version of notions of Bénabou [1], Penon [22] (see Johnstone's lemma A.2 in [18], Paré-Schumacher [21]). Notice that the definition does not involve the choice of a cleavage for the fibration.

1.13 Proposition *Locally internal categories over* $\mathbf{E}$ *are the same as* Span$\mathbf{E}$*-categories with restriction.*

1.14 Remark We will see later (in Section 4) that this correspondence extends to an equivalence of bicategories. At this stage we only prove a bijection (up to obvious notions of isomorphism).

Proof We give the basic constructions. When X ia a Span$\mathbf{E}$-category, a fibration $p : F \longrightarrow \mathbf{E}$ is obtained by taking for F the objects of X and the underlying function as the projection p on objects. *Hom* in F is given by:

$$F(x,y) = \{(f,\alpha) | f : u \to v, \alpha : 1_u \to X(y,x) \cdot f\}$$

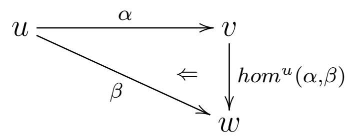
Composition and identities are defined as in the classical "Grothendieck construction". We obtain a projection from F to $\mathbf{E}$ by defining the effect of p on arrows as $p(f,\alpha) = f$. If we now assume that X has restrictions x_f, then a direct calculation shows that x_f satisfies the universal property of f^*x. Hence p is a fibration.
We can say more. Namely that $p : F \longrightarrow \mathbf{E}$ is a locally internal category. The enriched *Hom*, $X(x,y)$, is a span which represents $[x,y]$. The calculation involves the properties of the adjunctions $f \dashv f^\circ$ for maps.
Conversely, suppose we are given a locally internal category $p : F \longrightarrow \mathbf{E}$. A Span$\mathbf{E}$-category X is obtained by taking the objects of F as objects of X. The *Hom* in X is defined by an object which represents

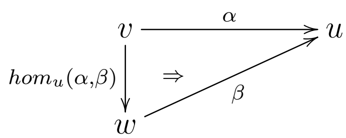
$$\mathrm{Span}(u,v)^{\mathrm{op}} \xrightarrow{[x,y]} \mathrm{Sets}$$

To show that X is a Span$\mathbf{E}$ category more calculations are required. It is then easy to prove that f^*x provides a restriction of x along f. ■

2 Modules

In this section we describe briefly the notion of module between B-categories, and some properties, for a general base bicategory B. The assumption we make here about the base bicategory is that it is locally finitely complete and cocomplete. Further we assume that it admits *right extensions* and *right liftings*, i.e. for each pair of arrows α and β as in the following pictures





there exists a right extension $hom^u(\alpha, \beta)$ (right lifting $hom_u(\alpha, \beta)$) characterized by the universal property

$$\frac{\gamma \to hom^u(\alpha, \beta)}{\gamma \cdot \alpha \to \beta} \quad (\text{resp. } \frac{\gamma \to hom_u(\alpha, \beta)}{\alpha \cdot \gamma \to \beta})$$

Observe that because of the existence of right adjoints to $\alpha \cdot -$ and $- \cdot \alpha$, composition with α on both sides preserves colimits.

2.1 Example (Monoidal categories). When B is a symmetric monoidal category the assumption of right extensions and right liftings amounts to requiring that B is closed.

2.2 Example (Span**E**). If **E** is a topos, Span**E** admits right extensions and right limits.

First observe that because Span**E** is a symmetric bicategory, the existence of right extensions implies the existence of right liftings (and conversely):

$$hom_u(\alpha, \beta) \cong (hom^u(\alpha^\circ, \beta^\circ))^\circ$$

Next, if $\alpha = g \cdot f^\circ$, extending along a composite we have

$$\begin{aligned} hom^u(g \cdot f^\circ, \beta) &\cong hom^u(g, \beta \cdot f) \\ &\cong \Pi_{g \times 1}(\beta \cdot f) \end{aligned}$$

When **E**=Sets, the formulae for right extensions and right liftings become

$$hom_u(\alpha, \beta)_{ik} = \Pi_j hom(\alpha_{ij}, \beta_{kj})$$
$$hom^u(\alpha, \beta)_{ik} = \Pi_j hom(\alpha_{ji}, \beta_{jk})$$

2.3 Definition *Suppose X and Y are B-categories. A* module $\phi : X \longrightarrow\!\!\!\!\!\!\!+\;\; Y$ *is the assignment of an arrow $\phi(x, y) : ex \to ey$ for every pair of objects, with an action of X on the left and of Y on the right, i.e. there are given 2-cells*

$$Y(y, y') \cdot \phi(x, y) \to \phi(x, y')$$
$$\phi(x, y) \cdot X(x', x) \to \phi(x', y)$$

satisfying the usual axioms of associativity, unity and mixed associativity.

When $\phi : X \nrightarrow Y$ and $\psi : Y \nrightarrow Z$ are modules, their *composition* $\psi \cdot \phi : X \nrightarrow Z$ is defined (if it exists) as follows: $(\psi \cdot \phi)(x, z)$ is the coequalizer in the category $B(ex, ez)$ of the two actions

$$\sum_{y',y''} \psi(y'', z) \cdot Y(y', y'') \cdot \phi(x, y') \rightrightarrows \sum_{y} \psi(y, z) \cdot \phi(x, y)$$

2.4 Remark A functor $F : X \to Y$ gives rise to two modules $F_* : X \nrightarrow Y$ and $F^* : Y \nrightarrow X$, defined by $F_*(x, y) = Y(Fx, y)$ and $F^*(y, x) = Y(y, Fx)$.

2.5 Example (Rings and modules). When B=Ab is the monoidal category of abelian groups, an internal category is just a ring and a module $\phi : R \nrightarrow S$ is a left-R-right-S module. Composition of such modules always exists and is the tensor product of modules.

A morphism $\alpha \to \beta$ of modules $\alpha, \beta : X \nrightarrow Y$ is given by a family of 2-cells

$$\alpha(x, y) \to \beta(x, y)$$

which is compatible with actions.

2.6 Remark Observe that, under our assumptions on the base B, composites of the type

$$X \overset{\phi}{\nrightarrow} A \overset{\psi}{\nrightarrow} Y$$

always exists when A is an internal category.

There are other special composites which always exist. In the situation

$$X \overset{F_*}{\nrightarrow} Y \overset{G^*}{\nrightarrow} Z$$

we have $(G^* \cdot F_*)(x, z) \cong Y(Fx, Gz)$.

2.7 Remark From the fact that composition preserves local colimits we can deduce that in the situation of the following diagram

$$X \overset{\alpha}{\nrightarrow} Y \overset{\beta}{\nrightarrow} Z \overset{\gamma}{\nrightarrow} W$$

if $\beta \cdot \alpha$ and $\gamma \cdot \beta$ exist then $(\gamma \cdot \beta) \cdot \alpha$ exists if and only if $\gamma \cdot (\beta \cdot \alpha)$ exists. In this case $(\gamma \cdot \beta) \cdot \alpha \cong \gamma \cdot (\beta \cdot \alpha)$.

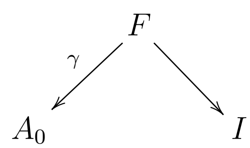
It follows that if A is an internal category and F is a functor $A \to X$, then F_* is left adjoint to F^* (the composites required to state this adjunction exist, the unit of the adjunction is the effect of F on arrows, the counit is composition in X).

2.8 Remark Reconsider now the restriction laws of section 1. By the previous remark we can deduce that the module $x_* \cdot f \cong X(x, -) \cdot f$ is left adjoint to $f^\circ \cdot x^* \cong f^\circ \cdot X(-, x)$ (because $x_* \dashv x^*$ and $f \dashv f^\circ$). This proves that the two isomorphisms relative to restrictions are equivalent, because $X(x_*, -) \dashv X(-, x^*)$.

2.9 Example Arrows $u \to v$ in the base are modules between discrete categories.

2.10 Example Modules $A \dashrightarrow u$ from an internal category A are just algebras for the monad A.

2.11 Example (Internal presheaves). In the Span$\mathbf{E}$ case, when A is an internal category, then modules $A \dashrightarrow I$ (I is the terminal object in $\mathbf{E}$) correspond exactly to internal presheaves. More precisely, the effect on objects $\gamma : F \to A_0$ of an internal presheaf gives rise to an arrow γ in Span $\mathbf{E}$:



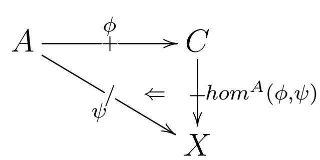
The effect on arrows $l : A_1 \times_{A_0} F \longrightarrow F$ provides a arrow $\Gamma \cdot A \to \Gamma$ of the monad A, and the preservation of composition and identities proves that the action l is that of an A-algebra Γ.

Now we describe right liftings and right extensions between modules.

2.12 Proposition *If A and C are internal categories and*

$$\psi : A \dashrightarrow X$$

is a module then $hom^A(\phi, \psi)$ exists



Proof An explicit calculation of $hom^A(\phi, \psi)(x)$ is provided by the equalizer of the following parallel pair of arrows in $B(w, ex)$ (w denotes the underlying object of C, v the underlying object of A):

$$hom^v(\phi, \psi x) \rightrightarrows hom^v(\phi, hom^v(a, \psi x))$$

■

An analogous statement and an analogous formula hold for right liftings.

2.13 Example (Rings and modules). When A, C, X are rings and ϕ, ψ are modules, then $hom^A(\phi, \psi)$ exists and it is the left-C-right-X module of A-linear maps.

We want now to represent modules. When A is an internal category, a new category PA can be obtained by taking as objects over u the modules $A \mathrel{\mapstochar\longrightarrow} u$, and as *hom* the right extension:

$$PA(\alpha, \beta) = hom^A(\alpha, \beta)$$

Example (PI). When $B = \text{Span}\mathbf{E}$, PI can be thought of as $\mathbf{E}$ itself regarded as a Span$\mathbf{E}$-category. Its objects over u are arrows $I \mathrel{\mapstochar\longrightarrow} u$, i.e. maps in $\mathbf{E}$ with codomain u. If $f : w \to u$ and $g : w' \to v$ are two objects, then

$$PI(f, g) = \Pi_{f\times 1}(1 \times g)$$

When $\mathbf{E} = \text{Sets}$, this formula can be written as follows:

$$PI(f, g)_{ij} = hom(f^{-1}(i), g^{-1}(j))$$

2.14 Example (Internal categories). When A is an internal category, then PA will be an example of functor category, namely $PA = (PI)^{A^{\mathrm{op}}}$ (see later, section 6).

2.15 Example When u is a discrete category, then the *hom* in Pu is directly given by the structure of the base:

$$Pu(\alpha, \beta) = hom^u(\alpha, \beta)$$

2.16 Proposition *PA represents modules.*

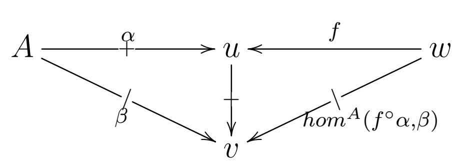
Proof To check the natural bijection

$$\frac{F : X \longrightarrow PA}{\hat{F} : A \mathrel{\mapstochar\longrightarrow} X}$$

it is enough to take $\hat{F}(a, x) = (Fx)a$. In particular the identity $X \mathrel{\mapstochar\longrightarrow} X$ corresponds to the Yoneda embedding $Yon : A \to PA$ which takes the only object of A into the representable module. ∎

2.17 Proposition *PA is a category with restrictions.*

Proof It is easy to check that, given two objects α and β in PA and a map $f : w \to u$ then $hom^A(\alpha, \beta) \cdot f \cong hom^A(f^\circ \cdot \alpha, \beta)$:



So $f^\circ \cdot \alpha$ is a restriction of α along f. ∎

We will see that restrictions are a particular type of indexed limits and we will prove a more general result about the existence of indexed limits in PA.

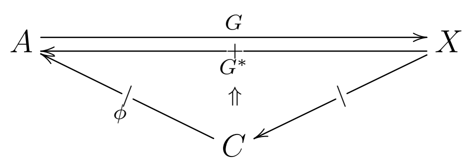
2.18 Example (Again PI). When $B = \text{Span(Sets)}$, with the calculation of the previous proposition we get that the restriction of the object $g : w \to u$ of PI along $f : v \to u$ is given by the pullback.

3 Completeness and cocompleteness

The limits and colimits we consider in a B-category are indexed by modules. The notion extends the analogous one given for categories based on a monoidal category (Street [23], Borceux-Kelly [10]). In detail

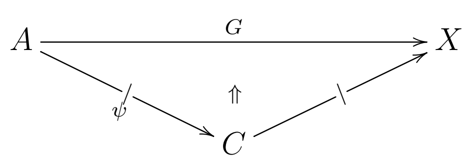
3.1 Definition *The* limit of G indexed by ϕ *(when it exists) is an object* $\{\phi, G\}$ *of* X *which represents the right lifting of* G^* *through* ϕ*, i.e.*

$$hom_A(\phi, G^*) \cong X(-, \{\phi, G\})$$



Analogously the colimit of G indexed by ψ *(when it exists) is an object* $\psi * G$ *which represents the right extension:*

$$hom^A(\psi, G_*) \cong X(\psi * G, -)$$



3.2 Remark Observe that, in the above definitions, the required lifting (or extension) might not exist. However they certainly do exist under our general assumptions on the base when A and C are internal categories. Notice further that the existence of the limit (or colimit) is not affected by the category structure on C. So generally we will take C to be discrete.

3.3 Definition X *is said to be* internal-complete (internal-cocomplete) *if it admits all limits (colimits) where the domain category is internal.*

3.4 Example (Restrictions). Restrictions give an example of limit (in this case usually called cotensor). It is enough to observe that when $A = v$ is a discrete category, $G = x$ is an object of X over v and $\phi = f$ is a map then $hom_A(f, x^*) \cong f^\circ \cdot x^* \cong f^\circ \cdot X(-, x)$.

Restrictions can also be calculated as colimits indexed by f°. We have dually:

$$hom^A(x_*, f^\circ) \cong x_* \cdot f \cong X(x, -) \cdot f$$

In more generality we have the following:

3.5 Proposition *When ψ has a left adjoint ϕ, then*

$$\{\phi, G\} \cong \psi * G \qquad \text{(if one exists)}$$

Proof It follows from the facts:

$$hom_A(\phi, G^*) \cong \psi \cdot G^* \quad \text{and} \quad hom^A(\psi, G_*) \cong G_* \cdot \phi$$

and the uniqueness of adjoints. ∎

3.6 Example (Cauchy sequences). Another example of indexed limit is obtained by considering the (usual) limit of a Cauchy sequence. Recall (Lawvere [20]) that a metric space is a category enriched over R^+ (non-negative real numbers, preordered by $\geq$ and monoidal with $+$).

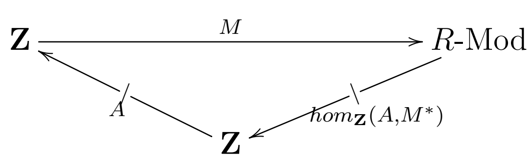
Let N be the null sequence $\{\frac{1}{n}\}$ of real numbers, considered as an R^+-category. Then a functor $x : N \to X$ is a Cauchy sequence in X dominated by this null sequence (and each Cauchy sequence is equivalent to such a sequence). Consider moreover the module $\phi : I \longrightarrow\!\!\!\!\!\!\!+\;\; N$ (I is the trivial R^+-category with one-object) whose components are $\phi(\frac{1}{n}) = \frac{1}{n}$.

Then a calculation gives

$$\lim_{n \to \infty} x_n = \{\phi, x\}$$

To test Cauchy-completeness of a metric space it is thus sufficient to check the existence of limits indexed by this particular module ϕ.

3.7 Example (R-modules). In the case $B = \text{Ab}$, consider the following diagram (where R is a ring):



M is an R-module and A is an abelian group. It is easy to check that

$$R - Mod(-, [A, M]) \cong hom_{\mathbf{Z}}(A, M)$$

where $[A, M]$ denotes the R-module of isomorphisms $A \to M$. Hence $[A, M]$ is the limit of M indexed by A.

In a similar way $A \otimes_{\mathbf{Z}} M$ is an instance of an indexed colimit:

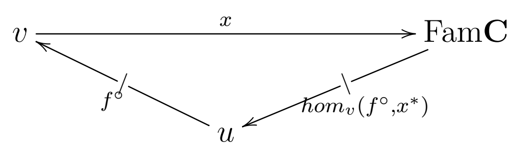
$$R - Mod(A \otimes_{\mathbf{Z}} M, -) \cong hom^{\mathbf{Z}}(A, M_*)$$

We consider the limit of a functor with a discrete domain $x : v \to X$, indexed by the opposite f° of a map (in the Span**E**-case) and we obtain the notion of *product indexed by* f.

3.8 Definition $\Pi_f x$ *is defined to be* $\{f^\circ, x\}$ *(when it exists).*

3.9 Example (Families). In the case $\mathbf{E}$ = Sets, $X = \mathrm{Fam}\mathbf{C}$ for a $\mathbf{C}$ with small products we have that $\{f^\circ, (x_j)_{j\in v}\}$ is the u-indexed family $(y_i)_{i\in u}$ given by

$$y_i = \Pi_{j\in f^{-1}i} x_j$$



Proof By applying the formula for right liftings given in section 2 we have

$$\begin{aligned} hom_v(f^\circ, \mathbf{C}(y,x)_{ki}) &\cong \Pi_j hom(f^\circ(i,j), \mathbf{C}(y_k, x_i)) \\ &\cong \Pi_{j\in f^{-1}i}\mathbf{C}(y_k, x_j) \cong \mathbf{C}(y_k, \Pi_{j\in f^{-1}i} x_j) \\ &\cong \mathrm{Fam}\mathbf{C}(y, \Pi_f x)_{ki} \end{aligned}$$

■

When $\mathbf{C}$ has small products, we have more:

3.10 Proposition Fam$\mathbf{C}$ *admits limits indexed by any arrow in the base, considered as a module. The limit* $\{\phi, x\}$ *can be computed by the formula:*

$$\{\phi, x\}_i = \Pi_j x_j^{\phi_{ij}}$$

where the exponents represents an iterated product.

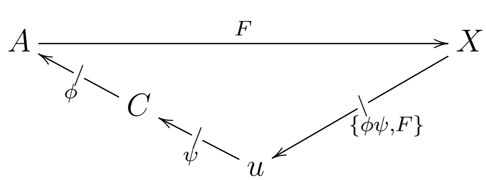
Proof It follows from the formulae for restriction and for products indexed by maps, using the following lemma. ■

3.11 Lemma *If* X *is an internal-complete* B*-category, then*

$$\{\phi\cdot\psi, F\} \cong \{\psi, \{\phi, F\}\}$$

Proof The proof relies entirely on universal properties of the right liftings involved:

$$hom_A(\phi\cdot\psi, F^*) \cong hom_C(\psi, hom_A(\phi, F^*))$$



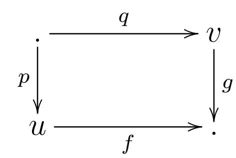
and essential uniqueness of their representing objects (existing because X is internal complete). ■

3.12 Remark A statement dual to that of the previous proposition holds true for colimits in Fam**C**. In this case the formula we get is:

$$(\psi * x)_j \cong \sum_i x_i \cdot \psi_{ij}$$

where $x_i \cdot \psi_{ij}$ denotes an iterated coproduct, and provided **C** is small-cocomplete.

3.13 Remark (Beck-Chevalley condition). Internal completeness of Span**E**-categories contains the Beck-Chevalley condition in the following sense: suppose we are given a pullback square in **E**



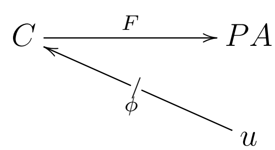
then the arrow $(p,q) = q \cdot p^\circ$ is isomorphic to $g^\circ \cdot f$ in Span**E**. Hence taken any $x : v \to X$ (with X internal-complete) we have

$$\{q \cdot p^\circ, x\} \cong \{g^\circ \cdot f, x\}$$

by the previous lemma we have: $f^* \cdot \Pi_g \cong \Pi_p \cdot q^*$.

3.14 Theorem *If A in an internal B-category, then PA is internal complete and cocomplete.*

Proof In the situation of the following diagram:



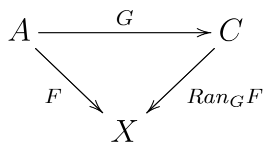
consider $\hat{F} : A \longrightarrow\!\!\!\!\!\!+\;\; C$ which is the module associated to F. We show that $\{\phi, F\} \cong hom_C(\phi, \hat{F})$. First observe that $\hat{F} = F^* \cdot Yon_*$, so we have

$$\begin{aligned} PA(-, hom_C(\phi, \hat{F})) &\cong PA(-, hom_C(\phi, F^* \cdot Yon_*)) \quad (Yon_* \text{ has a right adjoint}) \\ &\cong PA(-, hom_C(\phi, F^*) \cdot Yon_*) \quad (Yon \text{ is fully} - \text{faithful}) \\ &\cong hom_C(\phi, F^*) \end{aligned}$$

Analogously we can compute indexed colimits. ∎

By means of indexed limits and colimits it is possible also to express left and right extension of functors (when they exist). In the situation of the following diagram we have that, when X is internal-complete, the value of the right Kan extension $Ran_G F$ on the object c is given by

$$Ran_G F(c) \cong \{G^*(c,-), F\}$$



When X is internal-cocomplete, the left Kan extension $Lan_G F$ is similarly given by $Lan_G F(c) \cong G_*(-, c) * F$.

4 More about restrictions

We are now in a position to prove the equivalence announced in section 1 between Span**E**-categories with restrictions and locally internal categories. The morphisms between Span**E**-categories are just functors; the morphisms between locally internal categories are functors which preserve cartesian arrows and which commute with projections.
The main proposition is the following.

4.1 Proposition *Functors between B-categories preserve restrictions.*

Proof Let $f : u \to v$ be a map, $x : v \to X$ an object of X and $G : X \to Y$ a functor. We know that $x_f = \{f, x\}$. Because f has a right adjoint f° then

$$hom_v(f, x^*) \cong f^\circ \cdot x^* \cong f^\circ \cdot X(-, \{f, x\})$$

Hence

$$\begin{aligned} Y(-, F\{f, x\}) &\cong (F \cdot \{f, x\})^* \cong \{f, x\}^* \cdot F^* \\ &\cong f^\circ \cdot x^* \cdot F^* \cong Y(-, \{f, Fx\}) \end{aligned}$$

■

4.2 Remark The above proposition is part of a theorem of Street [27], in which absolute indexed limits are characterized as those whose indexing module has a right adjoint.

4.3 Proposition *The category of* Span**E***-categories with restrictions is equivalent to the category of locally internal categories.*

Proof Starting with a functor between Span**E**-categories $H : X \to Y$ we obtain a functor between the corresponding fibrations $\phi : F_X \to F_Y$ as follows. The effect of ϕ on objects is the same as that of H. Given an arrow $(f, \alpha) : 1_u \to X(x_2, x_1) \cdot f$ in X, then $\phi(f, \alpha)$ is

$$1_u \to X(x_2, x_1) \cdot f \to Y(Hx_2, Hx_1) \cdot f$$

where the second arrow is the effect of H on arrows. It is immediate to check that ϕ is a functor which commutes with the projections of F_X and F_Y. Moreover ϕ preserves cartesian arrows because H preserves restrictions.

Conversely, suppose we are given a functor $\phi : F \to G$ between locally internal categories. We obtain a functor H between the corresponding Span**E**-categories $X_F \to X_G$ as follows. The effect on objects is obvious. For each pair of objects x_1, x_2 in X_F let us consider their *hom* in Span**E**, i.e. the span $(f, g) = X_F(x_1, x_2)$. Since F is a locally internal category, then corresponding to the identity $X_F(x_1, x_2) \to X_F(x_1, x_2)$ there is an arrow $f^*x_1 \to g^*x_2$ in F. By applying the functor ϕ we have an arrow $\phi(f^*x_1) \to \phi(g^*x_2)$. Since ϕ preserves cartesian arrows we get an arrow $f^*(Hx_1) \to g^*(Hx_2)$. Also G is a locally internal category hence such arrows correspond to arrows

$$X_F(x_1, x_2) = (f, g) \to X_G(Hx_1, Hx_2)$$

in Span**E**. We have thus described the effect of the functor H on arrows. ∎

We will now describe how to adjoin freely restrictions to a category. The construction is as follows (see also Street [25]). Given a category X, the objects of LX over v are pairs (x, h) where $h : v \to u$ is a map of B and x is an object of X over u. The *hom* is given by

$$LX((x, h), (y, k)) = k^\circ \cdot X(x, y) \cdot h$$

Restrictions in LX are given by $(x, h)_f = (x, h \cdot f)$.

We have a functor $\Delta : X \to LX$ given by $x| \to (x, 1)$. That L is a functor results by the following proposition.

4.4 Proposition *LX is the free category with restrictions generated by X.*

Proof Suppose $F : X \to Y$ is any functor and Y has restrictions. Then we can define $G : LX \to Y$ by $G(x, h) = (Fx)_h$ and check that $G \cdot \Delta \cong F$. So

$$B - cat(X, Y) \cong B - cat(LX, Y)$$

∎

4.5 Example (Internal categories). Reconsider the notion of an internal functor. It is a map $f : A \to C$ which is a monad-map, i.e. it is endowed with a 2-cell $f \cdot A \to C \cdot f$. Now f becomes a functor $A \to LC$: it is enough to give an object of LC, namely $(*, f)$, where $*$ is the only object of C.
In fact we have:

$$Int\ Cat(A, C) \cong B - Cat(A, LC) \cong B - Cat(LA, LC)$$

4.6 Example (Internal full subcategory) When X is a B-category with restrictions and x is an object of X, we can consider the *internal full subcategory* determined by x by taking a one-object category with the same underlying of x and $hom(x, x)$ as *hom*.

When $B =$ Span**E** this is Penon's notion (Johnstone [18]). The original notion, due to Bénabou, is concerned with $X = PI$: given $f : v \to u$ in **E**, consider it as an object of PI. $\text{Full}_{\mathbf{E}}(f)$ is the internal category determined by the object f.

In general: it is trivial to verify that if A is an internal category and $f : LA \longrightarrow X$ is any functor, the induced functor $A \to X$ determines just one object x of X, and f can be uniquely factored as $f = h \cdot Lg$

$$La \xrightarrow{Lg} LC \xrightarrow{h} X$$

(where C is the internal full subcategory associated to x).

5 Closed bicategories

5.1 Question *What should a closed bicategory be?*

Thinking of objects of B as indexing types for families, it is necessary to consider several variables at the same time, and to interchange or separate them. The tool that enables us to accomplish this aim is a product in the base. With this new structure the analogy with symmetric monoidal closed categories becomes even more evident and indeed most of the classical theory of categories enriched over a symmetric monoidal closed category extends in a natural way to closed bicategories as described in this section.

5.2 Definition *A* tensor product *in B is a homomorphism of bicategories*

$$\otimes : B \times B \longrightarrow B$$

which is associative, symmetric and has an identity I.

5.3 Remark The properties of associativity, symmetry and identity asked for the $\otimes$ are intended up to equivalence of objects in B. In this paper we will not enter into the necessary coherence conditions, but we will rely on experience with the example Span**E**.

The following notion extends the notion of compact closed category introduced by Kelly ([15], p. 102).

5.4 Definition *A bicategory with a tensor product $\otimes : B \times B \to B$ is said to be* compact closed *when for each object v there exists an object v° and there are given isomorphisms of categories (natural in u and w and preserved by tensoring with an object) called* interchange of variables*:*

$$\frac{u \otimes v \to w}{u \to v^\circ \otimes w} \quad \text{and} \quad \frac{u \to v \otimes w}{u \otimes v^\circ \to w}$$

(The above notation just indicates the bijection of the isomorphisms on objects. Either one of these isomorphisms implies the other, see Kelly-Laplaza [17]).

5.5 Remark We usually denote arrows which correspond under the interchange of variables with the same symbol. If there is ambiguity the correspondence will be denoted by $(\hat{\ })$.

5.6 DEFINITION *A* closed bicategory *is a bicategory B which:*
(i) is locally finitely complete and cocomplete,
(ii) has right extensions and right liftings,
(iii) is endowed with a tensor product with respect to which it is compact closed.

5.7 REMARK From the properties of the $\otimes$ we have $(u^\circ)^\circ = u$ and $I^\circ = I$. In fact the correspondence $u \mapsto u^\circ$ extends to an involutory homomorphism $B^{\mathrm{op}} \to B$ (where B^{op} has just arrows reversed). Then the isomorphisms of the interchange of variables are natural also in v.

5.8 EXAMPLE (Monoidal closed categories). As already remarked, any symmetric monoidal closed category provides an example of a closed bicategory (with just one object); in this case $u^\circ = u$ and $\otimes$ is composition.

5.9 EXAMPLE (V-mod). When V is a symmetric monoidal closed category which admits small limits and colimits, then the category V-mod whose objects are V-categories and whose arrows are modules is a closed bicategory. It is known that V-mod has right extensions and right liftings. The $\otimes$ is given by the ordinary tensor of V-categories:

$$(A \otimes B)((a,b),(a',b')) = A(a,a') \otimes B(b,b')$$

On arrows the tensor product is given by

$$(\phi \otimes \psi)((a,c),(b,d)) = \phi(a,b) \otimes \psi(c,d)$$

In this case A° is the usual opposite category and the isomorphism

$$\frac{A \otimes B \longrightarrow\!\!\!\!\!\!\!+\;\; C}{A \longrightarrow\!\!\!\!\!\!\!+\;\; B^\circ \otimes C}$$

is verified by observing that both modules correspond to $A \otimes B \otimes C^\circ \longrightarrow\!\!\!\!\!\!\!+\;\; I$.

5.10 EXAMPLE (Relations). When $\mathbf{E}$ is a regular category, consider the category Rel$\mathbf{E}$ of relations of $\mathbf{E}$. The tensor product is the usual product of relations, $u^\circ = u$ and the interchange of variables is easily verified.

5.11 EXAMPLE (Span$\mathbf{E}$). Span$\mathbf{E}$ provides another example of a closed bicategory. We have already remarked the existence of right extensions and right liftings. The product in Span$\mathbf{E}$ is given by the product in $\mathbf{E}$ for objects, and on morphisms as follows: if $\alpha : u \longrightarrow\!\!\!\!\!\!\!+\;\; v$ and $\beta : u' \longrightarrow\!\!\!\!\!\!\!+\;\; v'$ are spans, then $\alpha \otimes \beta$ is the matrix $u \times u' \longrightarrow\!\!\!\!\!\!\!+\;\; v \times v'$ given by $\alpha_{ij} \times \beta_{kl}$ $(i \in u,\ j \in u',\ k \in v,\ l \in v')$.
In this case $u^\circ = u$ and the interchange of variables is satisfied because all the arrows involved are equal as maps in $\mathbf{E}$ with codomain $u \times v \times w$.

5.12 DEFINITION *When X is a B-category, the* opposite category X^{op} *has the same objects as X, underlying object equal to $(ex)^\circ$ and hom given by $X^{\mathrm{op}}(x,y) = X(y,x)^\circ$.*

5.13 Definition *If X and Y are B-categories, the* tensor product category $X \otimes Y$ *is defined as follows: the objects are the pairs (x, y) with x in X and y in Y, the underlying object of (x, y) is $ex \otimes ey$ and $(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$.*

5.14 Example (Span**E**). The tensor product of internal categories is the usual cartesian product.

5.15 Example (B-Mod). When B is, in addition, locally small complete and cocomplete, we can consider the bicategory B-Mod whose objects are small B-categories and whose arrows are modules. By extending directly the example of V-Mod given in this section, we have that B-Mod is a closed bicategory with respect to the opposite operation $(\ )^{\mathrm{op}}$ and to tensor product of categories.

When B is just a closed bicategory, we can still form the closed bicategory of internal categories and modules.

5.16 Example (Families). Consider Fam**C** and Fam**D**, i.e. the categories of families of given categories **C** and **D**. Then Fam**C** $\otimes$ Fam**D** is given by the families $(c_i, d_j)_{(i,j) \in u \otimes v}$ with the obvious *hom*.

There exists also the cartesian product Fam**C** $\times$ Fam**D**: the objects are families $(c_i, d_i)_{i \in u}$. It is easy to see that the cartesian product has restrictions (given component wise). The relationship with the tensor product is as follows:

$$L(\mathrm{Fam}\mathbf{C} \otimes \mathrm{Fam}\mathbf{D}) \cong \mathrm{Fam}\mathbf{C} \times \mathrm{Fam}\mathbf{D}$$

We describe the above equivalence on objects: from an object $((f, g) : w \to u \otimes v, (c_i, d_j)_{(i,j) \in u \otimes v})$ we get the object $(c_{fk}, d_{gk})_{k \in w}$. Conversely, given $(c_i, d_i)_{i \in u}$ in Fam**C** $\times$ Fam**D**, consider the diagonal $\Delta : u \to u \otimes u$ ($= u \times u$ because $B = \mathrm{Span(Sets)}$), and take the object $(\Delta : u \to u \otimes u, (c_i, d_j)_{i \times j \in u \times v})$ in L(Fam**C** $\otimes$ Fam**D**).

5.17 Remark We can now consider the category PI for any closed base B. In the case $B = \mathrm{Span}\mathbf{E}$ we have interpreted PI as **E** itself, as a Span**E**-category (see section 3). But also when $B = V$ is a monoidal closed category, PI is V itself considered as a V-category. Hence we write $PI = B$, and we can give a *Hom functor* for any category X.

5.18 Definition *$Hom_X : X^{\mathrm{op}} \otimes X \to B$ is the B-functor which takes (x, x') to $X(x, x')$ considered as an arrow $I \to ex \otimes ex'$.*

Further, modules can be represented as functors with codomain B.

5.19 Proposition *There is an isomorphism of categories*

$$\frac{\phi : X \longrightarrow\!\!\!\!\!\!\!+\ \ Y}{F : X^{\mathrm{op}} \otimes Y \to B}$$

Proof Arrows $\phi(x, y) : u \to v$ correspond bijectively to arrows $F(x, y) : I \to u^{\circ} \otimes v$. ■

6 Ends and functor categories

We now wish to investigate further the internal completeness of categories based on a closed bicategory, with the aim of defining the functor category X^A, when A is an internal category. This can be done by suitably extending the end formula by Day-Kelly [13]. Consider the module

$$\hat{1}_A : I \to A^{\text{op}} \otimes A$$

corresponding to the identity

$$I \otimes A \cong A \to A$$

Definition. Let A be an internal category. Given a functor

$$T : A^{\text{op}} \otimes A \to X$$

the *end* of X (if it exists) is the limit $\{\hat{1}_A, T\}$. We use the notation

$$\{\hat{1}_A, T\} = \int_A T$$

6.1 Example (Families). Consider the category Fam**C**, where **C** is an ordinary small-complete category. Let A be any category internal to Span(Sets), i.e. an ordinary small category, and consider

$$T : A^{\text{op}} \otimes A \to \text{Fam}\mathbf{C}$$

We prove that $\int_A T$ exists and give a formula to compute it. Let u be the set of objects of A. First consider the two arrows $\hat{1}_A$ and $\hat{1}_u$ in the base

$$I \rightrightarrows u^\circ \otimes u$$

We can compute separately the limits $\{\hat{1}_u, T\}$ and $\{\hat{1}_A, T\}$ regarding T as a functor from the discrete category underlying $A^{\text{op}} \otimes A$, i.e.

$$T : u^\circ \otimes u \to \text{Fam}\mathbf{C}$$

We have

$$\{\hat{1}_u, T\} = \Pi_{i\in u} T(i,i) \quad \text{and}$$
$$\{\hat{1}_A, T\} = \Pi_{i,j} T_{ij}^{A^{ij}}$$

(see the relative formulas in section 3). Such products exist because **C** is small complete. There are two arrows

$$\{\hat{1}_A, T\} \rightrightarrows \{\hat{1}_u, T\}$$

assigned by combining the effect of T on arrows, namely $T(1,f)$ and $T(f,1)$: $\int T$ is their equalizer in $\mathbf{C}$

$$\int T \rightarrowtail \{\hat{1}_A, T\} \rightrightarrows \{\hat{1}_u, T\}$$

Remark. We can also consider ends with extra-variables, i.e. given a functor

$$T : D \otimes A^{\mathrm{op}} \otimes A \longrightarrow X$$

we denote by $\int_A T : D \to X$ the limit $\{\hat{1}_D \otimes \hat{1}_A, T\}$.

Consider the module $\phi : I \nrightarrow A$, where A is an internal category (we take I for simplicity, the same argument works as well for a general internal D). Then, by the naturality of the interchange of variables (section 5), ϕ factorizes as

$$I \overset{\hat{1}_A}{\nrightarrow} A^{\mathrm{op}} \otimes A \overset{\phi^\circ \otimes 1}{\longrightarrow} I \otimes A \cong A$$

6.2 Proposition *(End formula for limits).*

$$\{\phi, F\} \cong \int_A \{\phi^\circ \otimes 1_A, F\}$$

if the right hand side exists.

Proof Using the lemma on iterated limits (section 3) we see that:

$$\begin{aligned} \{\phi, F\} &\cong \{(\phi^\circ \otimes 1_A) \cdot \hat{1}_A, F\} \cong \{\hat{1}_A, \{(\phi^\circ \otimes 1_A), F\}\} \cong \\ &\cong \int_A \{\phi^\circ \otimes 1_A, F\} \end{aligned}$$

■

6.3 Remark Let u be the underlying object of A, and consider limits of the type $\{\phi^\circ \otimes 1_A, F\}$:

$$\begin{array}{ccccc} I \otimes u & \longrightarrow & I \otimes A & \overset{F}{\longrightarrow} & X \\ {\scriptstyle \phi^\circ \otimes 1_u} \uparrow & & {\scriptstyle \phi^\circ \otimes 1_A} \uparrow & & \\ u^\circ \otimes u & & A^{\mathrm{op}} \otimes A & & \end{array}$$

To calculate $\{\phi^\circ \otimes 1_A, F\}$ we may first calculate $\{\phi^\circ \otimes 1_u, F\} : u^\circ \otimes u \to X$, and then there is a canonical way to extend it to a functor $A^{\mathrm{op}} \otimes A \to X$.
To see this, notice that, by the interchange of variables,

$$hom_{I \otimes A}(\phi^\circ \otimes 1_A, F^*) : X \nrightarrow A^{\mathrm{op}} \otimes A$$

can be calculated by means of $hom_I(\phi^\circ, F^*) : A \otimes X \to A$ which does not involve the category structure of A.

6.4 Example (Families). In Fam**C**, by applying the second proposition of section 3 and the above remark, we have

$$\{\phi^\circ \otimes 1, F\}_{ij} = F_j^{\phi_i}$$

Now, by the end formula for limits, we have:

6.5 Proposition *(Fubini theorem). Given $T : A^{\mathrm{op}} \otimes A \otimes D^{\mathrm{op}} \otimes D \to X$ we have*

$$\int_A \int_D T \cong \int_{A \otimes D} T \cong \int_D \int_A T \quad \text{(if any exists)}$$

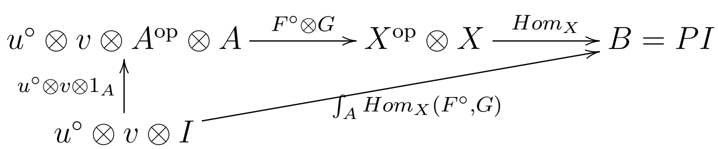
Proof Just observe that

$$\hat{1}_{A\otimes D} \cong (1_{A^{\mathrm{op}}\otimes A} \otimes \hat{1}_D) \cdot \hat{1}_A \cong (\hat{1}_A \otimes 1_{D^{\mathrm{op}}\otimes D}) \cdot \hat{1}_D$$

and apply the lemma on iterated limits (section 3). ∎

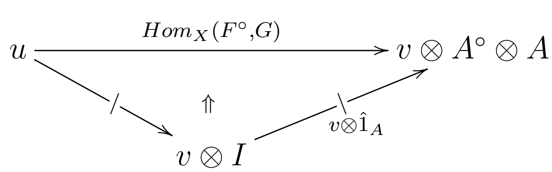
We now introduce *functor categories*. Suppose X is any B-category and A is internal. The functor category X^A has objects over u the functors $F : u \otimes A \longrightarrow X$. The *hom* in X^A is defined by

$$X^A(F, G) = \int_A Hom_X(F^\circ, G)$$



Observe that $\int_A Hom_X(F^\circ, G)$ is an object of PI over $u^\circ \otimes v$, i.e. it corresponds to an arrow $u \longrightarrow v$.

To check that X^A is a B-category, remembering how limits are computed in PI (section 3) and using the interchange of variables (section 5), we see that $\int_A Hom_X(F^\circ, G)$ is obtained as a right lifting in the base:



The computation involves interchange of variables and standard arguments relative to right liftings.

6.6 Example (Internal presheaves). We have:

$$PA \cong B^{A^{\mathrm{op}}}$$

The correspondence on objects is the following: given an object $\phi : A \longrightarrow\!\!\!\!\!\!+\; u$ over u in PA, we get a module $I \longrightarrow\!\!\!\!\!\!+\; u \otimes A^{\mathrm{op}}$ by the interchange of variables of one-object B-categories. Hence an object (over u) $u \otimes A^{\mathrm{op}} \to PI = B$ in $B^{A^{\mathrm{op}}}$.

6.7 Example (Families). Consider a discrete category u. Then $(\mathrm{Fam}\mathbf{C})^u$ has objects over v the families indexed by $u \times v$. If x is over v and y is over w, the *hom* is given by:

$$(\mathrm{Fam}\mathbf{C})^u(x,y)_{jk} = \Pi_i \mathbf{C}(x_{ij}, y_{ik})$$

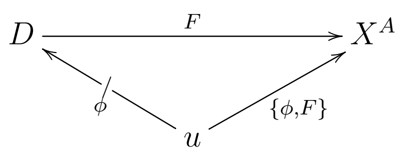
by applying the lifting formula in Span(Sets).

A calculation, long but straightforward, shows:

6.8 Theorem *If A is an internal category, then we have an isomorphism of categories:*

$$\frac{F : Y \longrightarrow X^A}{\bar{F} : A \otimes Y \longrightarrow X}$$

6.9 Remark It is easy to check that X^A has restriction whenever X has. More generally, if X is internal complete then X^A is internal complete and limits in X^A can be computed "pointwise", i.e. given



with D internal, then $\{\phi, F\} \cong \{\phi \otimes 1_A, \bar{F}\}$ where $\bar{F} : D \otimes A \to X$ corresponds to F.

Because the universal property of L and the fact (see section 4) that $L(A \otimes D) \cong LA \times LD$ for categories of families, we have:

6.10 Corollary *If A is internal and X, Y are categories of families, there is an equivalence of categories*

$$\frac{Y \to X^A}{LA \times Y \to X}$$

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